REPRESENTATIONS OF LITTLE q-SCHUR ALGEBRAS

JIE DU, QIANG FU AND JIAN-PAN WANG

Abstract. In [18] and [24], little q-Schur algebras were introduced as homomorphic images of the infinitesimal quantum groups. In this paper, we will investigate representations of these algebras. We will classify simple modules for little q-Schur algebras and classify semisimple little q-Schur algebras. Moreover, through the classification of the blocks of little q-Schur algebras for $n = 2$, we will determine little q-Schur algebras of finite representation type in the odd roots of unity case.

1. Introduction

The q-Schur algebras are certain finite dimensional algebras which were used by Jimbo in the establishment of the quantum Schur–Weyl reciprocity [33, Prop. 3] and were introduced by Dipper and James in [7],[8] in the study the representations of Hecke algebras and finite general linear groups. Using a geometric setting for q-Schur algebras, Belinson, Lusztig and MacPherson [1] reconstructed (or realized) the quantum enveloping algebra $\mathbf{U}(n)$ of $\mathfrak{gl}_n$ as a limit of q-Schur algebras over $\mathbb{Q}(v)$. This results in an explicit description of the epimorphism ζ_r from $\mathbf{U}(n)$ to the q-Schur algebra $\mathbf{U}(n,r)$ for all $r \geqslant 0$. Restriction induces an epimorphism from the Lusztig form $U_{\mathcal{Z}}(n)$ over $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ to the integral q-Schur algebra $U_{\mathcal{Z}}(n,r)$ [15] and, in particular, an epimorphism from $U_k(n)$ to $U_k(n,r)$ by specializing the parameter to any root of unity in a field k. The little q-Schur algebras $\tilde{u}_k(n,r)$ are defined as the homomorphic images of the finite dimensional Hopf subalgebra $\tilde{u}_k(n)$ of $U_k(n)$ under ζ_r. The structure of these algebras are investigated by the authors ([18],[24]). For example, through a BLM type realization for $\tilde{u}_k(n)$, various bases for $\tilde{u}_k(n,r)$ were constructed and dimension formulas were given. This paper is a continuation of [18] and [24].

It should be point out that, through the coordinate algebra approach, Doty, Nakano and Peters in [10] defined infinitesimal Schur algebras, closely related to the Frobenius kernels of an algebraic group over a field of positive characteristic. A theory for the quantum version of the infinitesimal Schur algebras was studied by Cox in [3] and [5]. The relation between the algebra structures of the infinitesimal and the little q-Schur algebra was investigated in [23]. Indeed, we see that the relation between the little and the infinitesimal q-Schur algebra is similar to that between the h-th Frobenius kernel G_h and the corresponding Jantzen subgroup G_hT. However, there is a subtle difference between infinitesimal q-Schur algebras and little q-Schur a1gebras.

Date: June 24, 2011.

Supported by the Natural Science Foundation of China (Grant No. 10971154), the Program NCET and the UNSW FRG.

Suppose ε is an l'-th root of 1 in a field k and define $l = l'$ if l' is odd, and $l = l'/2$, if l' is even. The parameter involved for defining G_h, G_hT and the infinitesimal q-Schur algebras is $q = \varepsilon^2$ which is always an l-th root of 1. So their representations are independent of l' (cf. [12, 3.1] or 4.3 below). However, the parameter used for defining little q-Schur algebras is ε which is a square root of q.[1] The structures and representations of $\tilde{u}_k(n)$ and little q-Schur algebras *do* depend on l' and are quite different (cf. [35, 5.11] or 5.2 below). In fact, the interesting case is the even case, where simple representations of $\tilde{u}_k(n)$ are indexed by $(\mathbb{Z}_{l'})^n$ and not every simple module can be obtained as a restriction of a simple module of $U_k(n)$ with an l-restricted highest weight. In contrast with the algebraic group case, this is a kind of "quantum phenomenon". Moreover, it should be noted that the representation theory of quantum enveloping algebras at the even roots of unity has found some new applications in the conformal field theory (or the theory of vertex operator algebras). See, e.g., [27] and [30].

We will show that every simple module of $\tilde{u}_k(n)$ is a restriction of a simple G_1T-module. To achieve this, we first classify simple $\tilde{u}_k(n,r)$-modules through the "sandwich" relation $u_k(n,r)_1 \subseteq \tilde{u}_k(n,r) \subseteq s_k(n,r)_1$ given in (4.1.2). By introducing the baby transfer map (cf. [36]), we will see a simple $\tilde{u}_k(n,r)$-module for $n \geqslant r$ and l' odd is either an inflation of a simple $\tilde{u}_k(n,r-l')$ via the module transfer map or a lifting of a simple module of the Hecke algebra via the q-Schur functor. Main results of the paper also include the classifications of semisimple little q-Schur algebras and, when l' is odd, the finite representation type of little q-Schur algebras.

We organize the paper as follows. We recall the definition for the infinitesimal quantum groups $\tilde{u}_k(\mathfrak{g})$ associated with a simple Lie algebra $\mathfrak{g}$ of a simply-laced type and $\tilde{u}_k(n)$ associated with $\mathfrak{gl}_n$ in §2. In §3, we shall study the baby Weyl module for infinitesimal quantum group $\tilde{u}_k(\mathfrak{g})$. We shall prove in 3.2 that for the restricted weight, the corresponding baby Weyl module is equal to the Weyl module. This is a well-known fact. Furthermore, using this result we can give another proof of [34, 7.1(c)(d)]. In §4, we shall recall some results about the little and the infinitesimal q-Schur algebra and establish the sandwich relation mentioned above. Moreover, classifications of simple G_h- and G_hT-modules from [12], [3] and [5] will be mentioned. In §5, we shall study the baby Weyl module for the infinitesimal quantum group $\tilde{u}_k(n)$ and give the classification of simple module for the little q-Schur algebra $\tilde{u}_k(n,r)$. The baby transfer maps are discussed in §6. In §7, we shall classify semisimple little q-Schur algebras, while in §8 we classify the finite representation type of little q-Schur algebras through the classification of the blocks of little q-Schur algebras for $n = 2$. Finally, in an Appendix, we show that the epimorphism from $U_{\mathcal{Z}}(n)$ onto $U_{\mathcal{Z}}(n,r)$ remains surjective when restricted to the Lusztig form $U_{\mathcal{Z}}(\mathfrak{sl}_n)$ of the quantum $\mathfrak{sl}_n$. Thus, the results developed in §3 for $\mathfrak{sl}_n$ can be directly used in §5.

[1] For this reason, little q-Schur algebras should probably be more accurately renamed as little $\sqrt{q}$-Schur algebras.

Throughout, let v be an indeterminate and let $\mathcal{Z}=\mathbb{Z}[v,v^{-1}]$. Let k be a field containing a primitive l'th root ε of 1 with $l'\geqslant 3$. Let $l>1$ be defined by

$$l=\begin{cases} l' & \text{if } l' \text{ is odd,}\\ l'/2 & \text{if } l' \text{ is even.}\end{cases}$$

Thus, ε^2 is always a primitive lth root of 1. Specializing v to ε, k will be viewed as a $\mathcal{Z}$-module.

For a finite dimensional algebra A over k, let $\mathrm{Mod}(A)$ be the category of finite dimensional left A-modules. If B is a quotient algebra of A, then the *inflation functor* embeds $\mathrm{Mod}(B)$ into $\mathrm{Mod}(A)$ as a full subcategory.

2. Lusztig's infinitesimal quantum enveloping algebras

In this section, following [35], we recall the definition for the infinitesimal quantum group.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra associated with an indecomposable positive definite symmetric Cartan matrix $C=(a_{ij})_{1\leqslant i,j\leqslant n}$.

Definition 2.1. The quantum enveloping algebra of $\mathfrak{g}$ is the algebra $\mathbf{U}(\mathfrak{g})$ over $\mathbb{Q}(v)$ generated by the elements

$$E_i,\ F_i,\ \mathsf{K}_i,\ \mathsf{K}_i^{-1}\ (1\leqslant i\leqslant n)$$

subject to the following relations :

(QG1) $\mathsf{K}_i\mathsf{K}_j=\mathsf{K}_j\mathsf{K}_i,\ \mathsf{K}_i\mathsf{K}_i^{-1}=1$;
(QG2) $\mathsf{K}_iE_j=v^{a_{ij}}E_j\mathsf{K}_i,\ \mathsf{K}_iF_j=v^{-a_{ij}}F_j\mathsf{K}_i$;
(QG3) $E_iF_j-F_jE_i=\delta_{ij}\frac{\mathsf{K}_i-\mathsf{K}_i^{-1}}{v-v^{-1}}$;
(QG4) $E_iE_j=E_jE_i,\ F_iF_j=F_jF_i$, if $a_{ij}=0$;
(QG5) $E_i^2E_j-(v+v^{-1})E_iE_jE_i+E_jE_i^2=0$, if $a_{ij}=-1$;
(QG6) $F_i^2F_j-(v+v^{-1})F_iF_jF_i+F_jF_i^2=0$, if $a_{ij}=-1$.

Definition 2.1 implies immediately the following result.

Lemma 2.2. *There is a unique $\mathbb{Q}(v)$-algebra automorphism σ on $\mathbf{U}(\mathfrak{g})$ satisfying*

$$\sigma(E_i)=F_i,\ \sigma(F_i)=E_i,\ \sigma(\mathsf{K}_i)=\mathsf{K}_i^{-1}.$$

For any integers c,t with $t\geqslant 1$, let $[c]=\frac{v^c-v^{-c}}{v-v^{-1}}\in\mathcal{Z}$, $[t]^!=[1][2]\cdots[t]$, and

$$\begin{bmatrix} c\\ t\end{bmatrix}=\prod_{s=1}^{t}\frac{v^{c-s+1}-v^{-c+s-1}}{v^s-v^{-s}}\in\mathcal{Z}.$$

If we put $[0]^!=0=\left[\begin{smallmatrix} c\\ 0\end{smallmatrix}\right]$, then $\left[\begin{smallmatrix} c\\ t\end{smallmatrix}\right]=[c]!/[t]![c-t]!$ for $c\geqslant t\geqslant 0$, and $\left[\begin{smallmatrix} c\\ t\end{smallmatrix}\right]=0$ for $t>c\geqslant 0$.

Let $U_{\mathcal{Z}}(\mathfrak{g})$ (resp., $U^+_{\mathcal{Z}}(\mathfrak{g}),U^-_{\mathcal{Z}}(\mathfrak{g})$) be the $\mathcal{Z}$-subalgebra of $\mathbf{U}(\mathfrak{g})$ generated by the elements $E_i^{(N)}=E_i^N/[N]!$, $F_i^{(N)}=F_i^N/[N]!$, and $\mathsf{K}_j^{\pm1}$ $(1\leqslant i\leqslant n-1,\ 1\leqslant j\leqslant n,\ N\geqslant 0)$ (resp.,

$E_i^{(N)}$, $F_i^{(N)}$). Let $U^0_{\mathcal{Z}}(\mathfrak{g})$ be the $\mathcal{Z}$-subalgebra of $\mathbf{U}(\mathfrak{g})$ generated by all $\mathsf{K}_i^{\pm 1}$ and $\left[{\mathsf{K}_i;0 \atop t}\right]$, where for $t \in \mathbb{N}$ and $c \in \mathbb{Z}$,

$$\begin{bmatrix} \mathsf{K}_i; c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{\mathsf{K}_i \upsilon^{c-s+1} - \mathsf{K}_i^{-1} \upsilon^{-c+s-1}}{\upsilon^s - \upsilon^{-s}}.$$

Proposition 2.3. *The following identities hold in* $U_{\mathcal{Z}}(\mathfrak{g})$.

$$F_i^{(N)} F_j^{(M)} = \sum_{N-M \leqslant s \leqslant N} (-1)^{s+N-M} \begin{bmatrix} s-1 \\ N-M-1 \end{bmatrix} F_i^{(N-s)} F_j^{(M)} F_i^{(s)}, \tag{2.3.1}$$

$$F_j^{(M)} F_i^{(N)} = \sum_{N-M \leqslant s \leqslant N} (-1)^{s+N-M} \begin{bmatrix} s-1 \\ N-M-1 \end{bmatrix} F_i^{(s)} F_j^{(M)} F_i^{(N-s)}, \tag{2.3.2}$$

where $N > M \geqslant 0$ *and* $a_{ij} = -1$.

Proof. Applying the algebra automorphism σ given in 2.2 to [35, 2.5(a),(b)], we get the desired formulas. $\square$

Regarding the field k as a $\mathcal{Z}$-algebra by specializing υ to ε, we will write $[t]_\varepsilon$ and $\left[{c \atop t}\right]_\varepsilon$ for the images of $[t]$ and $\left[{c \atop t}\right]$ in k, and define, following [35], the k-algebras $U_k^+(\mathfrak{g})$, $U_k^-(\mathfrak{g})$, $U_k^0(\mathfrak{g})$, and $U_k(\mathfrak{g})$ by applying the functor $(\ ) \otimes_{\mathcal{Z}} k$ to $U^+_{\mathcal{Z}}(\mathfrak{g})$, $U^-_{\mathcal{Z}}(\mathfrak{g})$, $U^0_{\mathcal{Z}}(\mathfrak{g})$, and $U_{\mathcal{Z}}(\mathfrak{g})$. We will denote the images of E_i, F_i, etc. in $U_k(\mathfrak{g})$ by the same letters.

Let $\tilde{u}_k^+(\mathfrak{g})$, $\tilde{u}_k^-(\mathfrak{g})$, $\tilde{u}_k^0(\mathfrak{g})$, and $\tilde{u}_k(\mathfrak{g})$ be the k-subalgebras of $U_k(\mathfrak{g})$ generated respectively by the elements E_i $(1 \leqslant i \leqslant n)$, F_i $(1 \leqslant i \leqslant n)$, $\mathsf{K}_i^{\pm 1}$ $(1 \leqslant i \leqslant n)$, and E_i, F_i, $\mathsf{K}_i^{\pm 1}$ $(1 \leqslant i \leqslant n)$. By a proof similar to [18, Th. 2.5], $\tilde{u}_k(\mathfrak{g})$ can be presented by generators E_i, F_i, $\mathsf{K}_i^{\pm 1}$ $(1 \leqslant i \leqslant n)$ and the relations (QG1)–(QG6) together with

$$E_i^l = 0 = F_i^l, \quad \mathsf{K}_i^{2l} = 1.$$

The algebra $\tilde{u}_k(\mathfrak{g})$ is called *the infinitesimal quantum group* associated with $\mathfrak{g}$. When $l' = l$ is odd, we will also call the algebra $u_k(\mathfrak{g}) = \tilde{u}_k(\mathfrak{g})/\langle \mathsf{K}_1^l - 1, \cdots, \mathsf{K}_n^l - 1\rangle$, considered in [35], an infinitesimal quantum group.

For the reductive complex Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$, we now modify the definitions above to introduce infinitesimal quantum $\mathfrak{gl}_n$ which will be used to define little q-Schur algebras in §4.

Let $\mathbf{U}(n) = \mathbf{U}(\mathfrak{gl}_n)$ be the quantum enveloping algebra of $\mathfrak{gl}_n$ which is a slightly modified version of Jimbo [33]; see [38, 3.2]. It is generated by the elements E_i, F_i $(1 \leqslant i \leqslant n-1)$ and $K_i^{\pm 1}$ $(1 \leqslant i \leqslant n)$ subject to the relations as given in [18, Def. 2.1].

Let $\widetilde{K}_i = K_i K_{i+1}^{-1}$ for $1 \leqslant i \leqslant n-1$. Then the subalgebra $'\mathbf{U}(n)$ of $\mathbf{U}(n)$ generated by the E_i, F_i and $\widetilde{K}_i$ $(1 \leqslant i \leqslant n-1)$ is isomorphic to the quantum enveloping algebra $\mathbf{U}(\mathfrak{sl}_n)$. By identifying $\widetilde{K}_i$ with K_i, we will identify $'\mathbf{U}(n)$ with $\mathbf{U}(\mathfrak{sl}_n)$.

Following [38], let $U_{\mathcal{Z}}(n)$ (resp., $U^+_{\mathcal{Z}}(n)$, $U^-_{\mathcal{Z}}(n)$) be the $\mathcal{Z}$-subalgebra of $\mathbf{U}(n)$ generated by all $E_i^{(m)}$, $F_i^{(m)}$, K_i and $\left[{K_i;0 \atop t}\right]$, (resp., $E_i^{(m)}$, $F_i^{(m)}$). Let $U^0_{\mathcal{Z}}(n)$ be the $\mathcal{Z}$-subalgebra of $\mathbf{U}(n)$ generated by all K_i and $\left[{K_i;0 \atop t}\right]$. Replacing K_i by $\widetilde{K}_i$, we may defined integral forms $'U_{\mathcal{Z}}(n)$, which is identified with $U_{\mathcal{Z}}(\mathfrak{sl}_n)$, and define $'U^0_{\mathcal{Z}}(n)$ and $'U^{\pm}_{\mathcal{Z}}(n) = U^{\pm}_{\mathcal{Z}}(n)$ similarly.

Let $U_k(n)=U_{\mathcal{Z}}(n)\otimes_{\mathcal{Z}}k$ and $'U_k(n)={}'U_{\mathcal{Z}}(n)\otimes_{\mathcal{Z}}k$. Since $'U_{\mathcal{Z}}(n)$ is a pure $\mathcal{Z}$-submodule of $U_{\mathcal{Z}}(n)$ (see [15, Prop. 2.6]), $'U_k(n)$ is a subalgebra of $U_k(n)$ identified with $U_k(\mathfrak{sl}_n)$.

Following [35], let $\tilde{u}_k(n)$ be the k-subalgebra of $U_k(n)$ generated by the elements E_i, F_i, $K_i^{\pm 1}$ $(1\leqslant i\leqslant n)$. Let $\tilde{u}_k^+(n)$, $\tilde{u}_k^0(n)$, $\tilde{u}_k^-(n)$ be the k-subalgebra of $\tilde{u}_k(n)$ generated respectively by the elements E_i $(1\leqslant i\leqslant n-1)$, $K_j^{\pm1}$ $(1\leqslant j\leqslant n)$, F_i $(1\leqslant i\leqslant n-1)$. We shall denote the images of E_i, F_i, etc. in $U_k(n)$, $\tilde{u}_k(n)$ by the same letters. In the case of l' being an odd number, let

$$u_k(n)=\tilde{u}_k(n)/\langle K_1^l-1,\cdots,K_n^l-1\rangle.$$

Similarly, we can define $'\tilde{u}_k(n)$ etc. as subalgebras of $'U_k(n)$, which are identified with $\tilde{u}_k(\mathfrak{sl}_n)$ etc.

3. Baby Weyl modules

Following [32, 5.15] for $\mathbf{d}=(d_1,\cdots,d_n)\in\mathbb{N}^n$ the $\mathbf{U}(\mathfrak{g})$-module $V(\mathbf{d})=\mathbf{U}(\mathfrak{g})/I(\mathbf{d})$ is irreducible where

$$I(\mathbf{d})=\sum_{1\leqslant i\leqslant n}(\mathbf{U}(\mathfrak{g})E_i+\mathbf{U}(\mathfrak{g})F_i^{d_i+1}+\mathbf{U}(\mathfrak{g})(\mathsf{K}_i-v^{d_i})).$$

Let $x_0=1+I(\mathbf{d})\in L(\mathbf{d})$. Let $V_{\mathcal{Z}}(\mathbf{d})$ be the $U_{\mathcal{Z}}(\mathfrak{g})$-submodule of $V(\mathbf{d})$ generated by x_0. Let $V_k(\mathbf{d})=V_{\mathcal{Z}}(\mathbf{d})\otimes_{\mathcal{Z}}k$. This is the Weyl module of $U_k(\mathfrak{g})$ with highest weight $\mathbf{d}$. For convenience, we shall denote the image of x_0 in $V_k(\mathbf{d})$ by the same letter. We call the $\tilde{u}_k(\mathfrak{g})$-module $V_k'(\mathbf{d}):=\tilde{u}_k(\mathfrak{g})x_0$ the *baby Weyl module* of $U_k(\mathfrak{g})$ (or the Weyl module of $\tilde{u}_k(\mathfrak{g})$).

Lemma 3.1. *Let $N\geqslant 0$ be an integer.*

(1) *If $Y\in\tilde{u}_k(\mathfrak{g})$ is a monomial in the F_i's, then*

$$F_i^{(N)}Y=\sum_{s\geqslant 0}X_sF_i^{(s)},\quad \textit{for some } X_s\in\tilde{u}_k^-(\mathfrak{g}).$$

(2) *If $Y\in\tilde{u}_k(\mathfrak{g})$ is a monomial in the E_i's, then have*

$$YE_i^{(N)}=\sum_{s\geqslant 0}E_i^{(s)}X_s,\quad \textit{for some } X_s\in\tilde{u}_k^+(\mathfrak{g}).$$

Proof. We only prove (1). The proof of (2) is similar.

Assume $Y=F_{j_1}^{(M_1)}F_{j_2}^{(M_2)}\cdots F_{j_t}^{(M_t)}$ where $0\leqslant M_i<l$ for all i. We proceed by induction on t.

Suppose $t=1$ and $M<l$. If $N\leqslant M$, then $F_i^{(N)}F_j^{(M)}\in\tilde{u}_k^-(\mathfrak{g})$, where $j=j_1$. Hence the result follows by putting $X_0=F_i^{(N)}F_j^{(M)}$. We now assume $0\leqslant M<N$. If $a_{ij}=0$, then $F_i^{(N)}F_j^{(M)}=F_j^{(M)}F_i^{(N)}$ by the definition 2.1. If $a_{ij}=-1$, then, by (2.3.1),

$$F_i^{(N)}F_j^{(M)}=\sum_{N-M\leqslant s\leqslant N}(-1)^{s+N-M}\begin{bmatrix}s-1\\N-M-1\end{bmatrix}F_i^{(N-s)}F_j^{(M)}F_i^{(s)}=\sum_{N-M\leqslant s\leqslant N}X_sF_i^{(s)},$$

where $X_s=(-1)^{s+N-M}\left[\begin{smallmatrix}s-1\\N-M-1\end{smallmatrix}\right]F_i^{(N-s)}F_j^{(M)}$ for $N-M\leqslant s\leqslant N$. Note that if $N-M\leqslant s$ with $M<l$ then $N-s<l$. Hence, $X_s\in\tilde{u}_k^-(\mathfrak{g})$.

Assume now $t > 1$. Let $Y' = F_{j_1}^{(M_1)} F_{j_2}^{(M_2)} \cdots F_{j_{t-1}}^{(M_{t-1})}$. Then by induction, we have

$$F_i^{(N)} Y' = \sum_{s \geqslant 0} X_s F_i^{(s)},$$

where $X_s \in \tilde{u}_k^-(\mathfrak{g})$. The previous argument show that for each $s \geqslant 0$, we have

$$F_i^{(s)} F_{j_t}^{(M_t)} = \sum_{m \geqslant 0} Y_{s,m} F_i^{(m)}, \text{ for some } Y_{s,m} \in \tilde{u}_k^-(\mathfrak{g}).$$

Let $X'_m = \sum_{s \geqslant 0} X_s Y_{s,m} \in \tilde{u}_k^-(\mathfrak{g})$ for $m \geqslant 0$. Then

$$F_i^{(N)} Y = \sum_{m \geqslant 0} X'_m F_i^{(m)},$$

as required. □

Theorem 3.2. *For $\mathbf{d} = (d_1, \cdots, d_n) \in \mathbb{N}^n$ with $0 \leqslant d_i < l$ for all i, we have $V'_k(\mathbf{d}) = V_k(\mathbf{d})$.*

Proof. By the definition of $V_k(\mathbf{d})$ and $V'_k(\mathbf{d})$, we have $V_k(\mathbf{d}) = U_k^-(\mathfrak{g}) x_0$ and $V'_k(\mathbf{d}) = \tilde{u}_k^-(\mathfrak{g}) x_0$. Since $U_k^-(\mathfrak{g})$ is generated by the elements $F_i^{(N)}$ $(1 \leqslant i \leqslant n,\ N \geqslant 0)$, it is enough to prove $F_i^{(N)} \tilde{u}_k^-(\mathfrak{g}) x_0 \subseteq \tilde{u}_k^-(\mathfrak{g}) x_0$ for all $1 \leqslant i \leqslant n$ with $N \geqslant 0$. For a monomial Y of F_i in $\tilde{u}_k(\mathfrak{g})$, by 3.1(1), we have

$$F_i^{(N)} Y x_0 = \sum_{s \geqslant 0} X_s F_i^{(s)} x_0, \tag{3.2.1}$$

where $X_s \in \tilde{u}_k^-(\mathfrak{g})$ and $N \geqslant 0$. Since $0 \leqslant d_i < l$ and $F_i^{(d_i+1)} x_0 = 0$ for all i, we have $F_i^{(s)} x_0 = 0$ for $s \geqslant l$. By (3.2.1), we have

$$F_i^{(N)} Y x_0 = \sum_{0 \leqslant s < l} X_s F_i^{(s)} x_0 \subseteq \tilde{u}_k^-(\mathfrak{g}) x_0.$$

Hence the result follows. □

Following [34, 4.6], we say that a $U_k(\mathfrak{g})$-module V has type $\mathbf{1}$ if $V = \{v \in V \mid \mathsf{K}_i^l v = v \text{ for } i = 1, \cdots, n\}$. Let V be a $U_k(\mathfrak{g})$-module of type $\mathbf{1}$. For any $\mathbf{z} = (z_1, \cdots, z_n) \in \mathbb{Z}^n$, following [34, 5.2], we define the $\mathbf{z}$-weight space

$$V_{\mathbf{z}} = \left\{ x \in V \mid \mathsf{K}_i x = \varepsilon^{z_i} x,\ \begin{bmatrix} \mathsf{K}_i; 0 \\ l \end{bmatrix} x = \begin{bmatrix} z_i \\ l \end{bmatrix}_\varepsilon x \ for\ i = 1, \cdots, n \right\}.$$

Lemma 3.3. ([34, 4.2])*Let V be a $U_k(\mathfrak{g})$-module and let $x \in V$ be such that $\mathsf{K}_i x = \varepsilon^m x$ for some $m \in \mathbb{Z}$. Then for any $c, c' \in \mathbb{Z}$ we have $\begin{bmatrix} \mathsf{K}_i;c \\ l \end{bmatrix} x - \begin{bmatrix} \mathsf{K}_i;c' \\ l \end{bmatrix} x = \left(\begin{bmatrix} m+c \\ l \end{bmatrix}_\varepsilon - \begin{bmatrix} m+c' \\ l \end{bmatrix}_\varepsilon \right) x \in \mathbb{Z}x$.*

Using 3.3, we see that $\begin{bmatrix} \mathsf{K}_i;c \\ l \end{bmatrix} x = \begin{bmatrix} z_i+c \\ l \end{bmatrix}_\varepsilon x$, $x \in V_{\mathbf{z}}$, $c \in \mathbb{Z}$. Let $\alpha(i) = (a_{1i}, a_{2i}, \cdots, a_{ni}) \in \mathbb{Z}^n$, $(1 \leqslant i \leqslant n)$. We define a partial order on $\mathbb{Z}^n$ by $\mathbf{z} \leqslant \mathbf{z}' \Leftrightarrow \mathbf{z}' - \mathbf{z} = \sum_{i=1}^{n} c_i \alpha(i)$ for some $c_1, \cdots, c_n \in \mathbb{N}$. This is a partial order on $\mathbb{Z}^n$. It is clear that we have the following lemma.

Lemma 3.4. *Let V be a $U_k(\mathfrak{g})$-module of type $\mathbf{1}$. Then for $N>0$, we have*
(1) $E_i^{(N)}V_{\mathbf{z}} \subseteq V_{\mathbf{z}+N\alpha(i)}$
(2) $F_i^{(N)}V_{\mathbf{z}} \subseteq V_{\mathbf{z}-N\alpha(i)}$.

By [34, 6.2], for $\mathbf{d}=(d_1,\cdots,d_n)\in\mathbb{N}^n$, $V_k(\mathbf{d})$ has a unique maximal $U_k(\mathfrak{g})$-submodule $W_k(\mathbf{d})$. Let $L_k(\mathbf{d})=V_k(\mathbf{d})/W_k(\mathbf{d})$. Then $L_k(\mathbf{d})$ is a simple $U_k(\mathfrak{g})$-module. Similarly, the $\tilde{u}_k(\mathfrak{g})$-module $V_k'(\mathbf{d})$ has a unique maximal $\tilde{u}_k(\mathfrak{g})$ submodule $W_k'(\mathbf{d})$ by the proof of [35, 5.11]. Let $L_k'(\mathbf{d})=V_k'(\mathbf{d})/W_k'(\mathbf{d})$. Then $L_k'(\mathbf{d})$ is a simple $\tilde{u}_k(\mathfrak{g})$-module.

Let I^- be the ideal in $\tilde{u}_k^-(\mathfrak{g})$ spanned as a k-vector space by the nonempty words in F_i, $1\leqslant i\leqslant n$.

Lemma 3.5. *Assume $\mathbf{d}\in\mathbb{N}^n$ with $d_i<l$ for all i. Let x_0 be the highest weight vector of $V(\mathbf{d})$. Then we have*

$$I^-x_0=\sum_{\mathbf{z}<\mathbf{d}}V_k(\mathbf{d})_{\mathbf{z}}.$$

Proof. It is clear that $I^-x_0\subseteq\sum_{\mathbf{z}<\mathbf{d}}V_k(\mathbf{d})_{\mathbf{z}}$. Since $V_k(\mathbf{d})=kx_0\oplus\sum_{\mathbf{z}<\mathbf{d}}V_k(\mathbf{d})_{\mathbf{z}}$ and $V_k'(\mathbf{d})=kx_0\oplus I^-x_0$, by 3.2, we have $kx_0\oplus\sum_{\mathbf{z}<\mathbf{d}}V_k(\mathbf{d})_{\mathbf{z}}=kx_0\oplus I^-x_0$. Hence $\dim\sum_{\mathbf{z}<\mathbf{d}}V_k(\mathbf{d})_{\mathbf{z}}=\dim I^-x_0$, and the result follows. □

Lemma 3.6. *Assume $\mathbf{d}\in\mathbb{N}^n$ with $d_i<l$ for all i. Let x_0 be the highest weight vector of $V(\mathbf{d})$. Then, $E_i^{(N)}I^-x_0\subseteq I^-x_0$ whenever $N\geqslant l$.*

Proof. Let $Y=F_{j_1}^{M_1}F_{j_2}^{M_2}\cdots F_{j_s}^{M_s}$ where $0<M_j<l$ for all j. If $E_i^{(N)}Yx_0\notin I^-x_0$, then by 3.5 we have $E_i^{(N)}Yx_0\in V_k(\mathbf{d})_{\mathbf{d}}$. By 3.4, we have $N\alpha(i)=M_1\alpha(j_1)+\cdots+M_s\alpha(j_s)$. Since $\alpha(1),\cdots,\alpha(n)$ are linearly independent, we have $N=M_1+\cdots+M_s$ and $j_1=\cdots=j_s=i$. So $Y=F_i^N$. Since $F_i^l=[l]_\varepsilon!F_i^{(l)}=0$ and $N\geqslant l$, we have $Y=0$. This is a contradiction. □

The following result is given in [34, 7.1(c)(d)] when l' is odd.

Theorem 3.7. *Assume that $\mathbf{d}=(d_1,\cdots,d_n)\in\mathbb{N}^n$ with $0\leqslant d_i<l$ for all i. Then $L_k(\mathbf{d})=L_k'(\mathbf{d})$.*

Proof. By 3.2, it is enough to prove that $W_k(\mathbf{d})=W_k'(\mathbf{d})$. Also, by 3.2, the restriction of $W_k(\mathbf{d})$ to $\tilde{u}_k(\mathfrak{g})$ is a submodule of $V_k(\mathbf{d})=V_k'(\mathbf{d})$. Hence, by the maximality of $W_k'(\mathbf{d})$, we have $W_k(\mathbf{d})\subseteq W_k'(\mathbf{d})$. On the other hind, we consider the $U_k(\mathfrak{g})$-submodule V of $V_k(\mathbf{d})$ generated by $W_k'(\mathbf{d})$. We shall prove that $V\subseteq\sum_{\mathbf{z}<\mathbf{d}}V_k(\mathbf{d})_{\mathbf{z}}$. Since $W_k'(\mathbf{d})$ is a $\tilde{u}_k(\mathfrak{g})$-module, by 3.1(2), we have

$$\begin{aligned}U_k^+(\mathfrak{g})W_k'(\mathbf{d})&\subseteq\mathrm{span}\{E_{i_1}^{(N_1)}\cdots E_{i_s}^{(N_s)}W_k'(\mathbf{d})\mid s\geqslant0,\ N_i\geqslant l\ for\ all\ i\}\\&\subseteq\{E_{i_1}^{(N_1)}\cdots E_{i_s}^{(N_s)}I^-x_0\mid s\geqslant0,\ N_i\geqslant l\ for\ all\ i\}\\&\subseteq I^-x_0\qquad\text{(by 3.6)}.\end{aligned}$$

Hence, by 3.5 and 3.4, $V = U_k^-(\mathfrak{g})U_k^+(\mathfrak{g})W_k'(\mathbf{d}) \subseteq U_k^-(\mathfrak{g})I^-x_0 = U_k^-(\mathfrak{g})\sum_{\mathbf{z}<\mathbf{d}} V_k(\mathbf{d})_{\mathbf{z}} = \sum_{\mathbf{z}<\mathbf{d}} V_k(\mathbf{d})_{\mathbf{z}}$. So by the maximality of $W_k(\mathbf{d})$, we have $W_k'(\mathbf{d}) \subseteq V \subseteq W_k(\mathbf{d})$. The result follows. □

Remark 3.8. Note that, if l' is odd and $\mathbf{d} \in \mathbb{N}^n$ with $d_i < l$ for all i, then $L_k'(\mathbf{d}) = L_k(\mathbf{d})$ is also a $u_k(\mathfrak{g})$-module. So, by [35, 6.6], the $u_k(\mathfrak{g})$-module $L_k(\mathbf{d})$ ($\mathbf{d} \in \mathbb{N}^n,\ 0 \leqslant d_i < l$ for all i) give all simple $u_k(\mathfrak{g})$-modules.

4. The infinitesimal and little q-Schur algebras

In this section, we shall recall the definitions of the infinitesimal q-Schur algebra defined in [3] and [5] and the little q-Schur algebra defined in [18, 24].

For the moment, we assume that R is a ring and $q^{\frac{1}{2}} \in R$.

Following [6], let $A_q(n)$ be the R-algebra generated by the n^2 indeterminates c_{ij}, with $1 \leqslant i, j \leqslant n$, subject to the relations

$$
\begin{aligned}
&c_{ij}c_{il} = c_{il}c_{ij} && \text{for all } i, j, l,\\
&c_{ij}c_{rs} = qc_{rs}c_{ij} && \text{for } i > r \text{ and } j \leqslant s,\\
&c_{ij}c_{rs} = (q-1)c_{rj}c_{is} + c_{rs}c_{ij} && \text{for } i > r \text{ and } j > s.
\end{aligned}
$$

The algebra $A_q(n)$ has a bialgebra structure such that the coalgebra structure is given by

$$\Delta(c_{ij}) = \sum_{t=1}^{n} c_{it} \otimes c_{tj} \qquad \text{and} \qquad \epsilon(c_{ij}) = \delta_{ij}.$$

Let $A_q(n,r)$ denote the subspace of elements in $A_q(n)$ of degree r. Then $A_q(n,r)$ are in fact subcoalgebras of $A_q(n)$ for all r, and hence, $U_R(n,r) := A_q(n,r)^*$ is an R-algebra, which is call a *q-Schur algebra*.

Let $\Xi(n)$ be the set of all $n \times n$ matrices over $\mathbb{N}$. Let $\sigma : \Xi(n) \to \mathbb{N}$ be the map sending a matrix to the sum of its entries. Then, for $r \in \mathbb{N}$, the inverse image $\Xi(n,r) := \sigma^{-1}(r)$ is the set of $n \times n$ matrices in $\Xi(n)$ whose entries sum to r.

For $A \in \Xi(n)$, let

$$c^A = c_{1,1}^{a_{1,1}} c_{2,1}^{a_{2,1}} \cdots c_{n,1}^{a_{n,1}} c_{1,2}^{a_{1,2}} c_{2,2}^{a_{2,2}} \cdots c_{n,2}^{a_{n,2}} \cdots c_{1,n}^{a_{1,n}} c_{2,n}^{a_{2,n}} \cdots c_{n,n}^{a_{n,n}} \in A_q(n).$$

Then by [6] (see also [39]), the set $\{c^A \mid A \in \Xi(n,r)\}$ forms an R-basis for $A_q(n,r)$. Putting $\xi_A := (c^A)^*$, we obtain the dual basis $\{\xi_A \mid A \in \Xi(n,r)\}$ for the q-Schur algebra $U_R(n,r)$.

For $A \in \Xi(n,r)$, let

$$[A] = q^{\frac{-d_A}{2}} \xi_A \quad \text{where} \quad d_A = -\sum_{i<s,\, j>t} a_{i,j}a_{s,t} + \sum_{j>t} a_{i,j}a_{i,t}.$$

Then $\{[A]\}_{A \in \Xi(n,r)}$ forms also a basis for $U_R(n,r)$.

We now introduce the infinitesimal q-Schur algebras. Thus, we assume $R = k$ is a field of *characteristic* $p > 0$ and $q = \varepsilon^2 \in k$. Since ε is a primitive l'-th root of unity, q is *always* a primitive l-th root of unity.

Consider the following ideals in $A_q(n)$

$$\begin{aligned}
I_h &= \langle c_{ij}^{lp^{h-1}}, c_{ij}^{lp^{h-1}} - 1 \mid 1 \leqslant i \neq j \leqslant n \rangle, \\
\tilde{I}_h &= \langle c_{ij}^{lp^{h-1}}, c_{ii}^{l'p^{h-1}} - 1 \mid 1 \leqslant i \neq j \leqslant n \rangle, \text{ and} \\
J_h &= \langle c_{ij}^{lp^{h-1}} \mid 1 \leqslant i \neq j \leqslant n \rangle.
\end{aligned}$$

Clearly, $J_h \subseteq \tilde{I}_h \subseteq I_h$. Note that J_h is a graded ideal and, if l' is odd, then $l = l'$ and $I_h = \tilde{I}_h$.

Lemma 4.1. *The ideals I_h, $\tilde{I}_h$ and J_h are all coideals of $A_q(n)$.*

Proof. The assertion for J_h and I_h is well known, using [20, (3.4)]. More precisely, we have

$$\begin{aligned}
\Delta(c_{i,j}^{lp^{h-1}}) &= \sum_{1 \leqslant k \leqslant n} c_{i,k}^{lp^{h-1}} \otimes c_{k,j}^{lp^{h-1}} \in J_h \otimes A_q(n) + A_q(n) \otimes J_h, \quad (i \neq j) \\
\Delta(c_{i,i}^{lp^{h-1}} - 1) &= \sum_{k \neq i} c_{i,k}^{lp^{h-1}} \otimes c_{k,i}^{lp^{h-1}} + (c_{i,i}^{lp^{h-1}} - 1) \otimes c_{i,i}^{lp^{h-1}} + 1 \otimes (c_{i,i}^{lp^{h-1}} - 1) \\
&\in I_h \otimes A_q(n) + A_q(n) \otimes I_h.
\end{aligned}$$

If l' is odd, then $\tilde{I}_h = I_h$ is the coideal of $A_q(n)$. Now we assume l' is even. Then

$$\begin{aligned}
\Delta(c_{i,i}^{l'p^{h-1}} - 1) &= (\Delta(c_{i,i}^{lp^{h-1}}))^2 - 1 \otimes 1 \\
&= \left(\sum_{1 \leqslant k \leqslant n} c_{i,k}^{lp^{h-1}} \otimes c_{k,i}^{lp^{h-1}} \right)^2 - 1 \otimes 1 \\
&= \sum_{j \neq k} c_{i,j}^{lp^{h-1}} c_{i,k}^{lp^{h-1}} \otimes c_{j,i}^{lp^{h-1}} c_{k,i}^{lp^{h-1}} + \sum_{1 \leqslant k \leqslant n} c_{i,k}^{l'p^{h-1}} \otimes c_{k,i}^{l'p^{h-1}} - 1 \otimes 1 \\
&= \sum_{j \neq k} c_{i,j}^{lp^{h-1}} c_{i,k}^{lp^{h-1}} \otimes c_{j,i}^{lp^{h-1}} c_{k,i}^{lp^{h-1}} + \sum_{k \neq i} c_{i,k}^{l'p^{h-1}} \otimes c_{k,i}^{l'p^{h-1}} \\
&\qquad + (c_{i,i}^{l'p^{h-1}} - 1) \otimes c_{i,i}^{l'p^{h-1}} + 1 \otimes (c_{i,i}^{l'p^{h-1}} - 1) \\
&\in \tilde{I}_h \otimes A_q(n) + A_q(n) \otimes \tilde{I}_h.
\end{aligned}$$

Thus, $\tilde{I}_h$ is a coideal of $A_q(n)$. □

Now, by the above lemma, $A_q(n)/J_h$, $A_q(n)/I_h$ and $A_q(n)/\tilde{I}_h$ are all bialgebras, and $A_q(n)/J_h$ is graded. Let $A_q(n,r)_h$ be the subspace of $A_q(n)/J_h$ consisting of the homogeneous polynomials of degree r in the c_{ij}. Since $A_q(n,r)_h$ is a finite dimensional subcoalgebra of $A_q(n)/J_h$, its dual

$$s_k(n,r)_h = A_q(n,r)_h^*$$

is a finite dimensional algebra, which is called an *infinitesimal q-Schur algebra* in [3] and [5] (cf. [10]). There are two canonical maps

(4.1.1) $$\pi : A_q(n)/J_h \twoheadrightarrow A_q(n)/I_h \text{ and } \tilde{\pi} : A_q(n)/J_h \twoheadrightarrow A_q(n)/\tilde{I}_h.$$

Since $\pi(A_q(n,r)_h)$ and $\tilde{\pi}(A_q(n,r)_h)$ are all coalgebras, we may define the algebras

$$u_k(n,r)_h = (\pi(A_q(n,r)_h))^*, \quad \tilde{u}_k(n,r)_h = (\tilde{\pi}(A_q(n,r)_h))^*.$$

By definition, we see easily that

$$u_k(n,r)_h \subseteq \tilde{u}_k(n,r)_h \subseteq s_k(n,r)_h. \tag{4.1.2}$$

In the case of l' being an odd number, we have $u_k(n,r)_h = \tilde{u}_k(n,r)_h$. In general, we will use these inclusions together with results on simple modules of $u_k(n,r)_1$ and $s_k(n,r)_1$, which is stated in the next theorem, to determine all simple $\tilde{u}_k(n,r)_1$-modules in §5.

Remark 4.2. When l' is even, the coideal $\tilde{I}_h$ was not introduced in the literature, say, [6] or [3, 5]. The definitions of $u_k(n,r)_h$ and $s_k(n,r)_h$ are independent of l', while that of $\tilde{u}_k(n,r)$ depends on l'. We will establish below in 4.9 a connection between $\tilde{u}_k(n,r)_1$ and the little q-Schur algebra $\tilde{u}_k(n,r)$.

Let $D_q = \sum_{\pi\in\mathfrak{S}_n}(-1)^{\ell(\pi)}c_{1,1\pi}c_{2,2\pi}\cdots c_{n,n\pi} \in A_q(n)$ be the quantum determinant, where $\mathfrak{S}_n$ is the symmetric group and $\ell(\pi)$ is the length of π. Then the localization $A_q(n)_{D_q}$ is a Hopf algebra. Let $G = G_q(n)$ be the quantum linear group whose coordinate algebra is $k[G] := A_q(n)_{D_q}$. Following [12, §3.1,§3.2] (see also [3, 1.3] and [5]), let G_h be the h-th Frobenius kernel and G_hT, where $T = T_q(n)$ be the torus of G, be the corresponding "Jantzen subgroups". Then

$$k[G_h] := A_q(n)_{D_q}/\langle I_h\rangle \cong A_q(n)/I_h \quad \text{and} \quad k[G_hT] := A_q(n)_{D_q}/\langle J_h\rangle, \tag{4.2.1}$$

and $A_q(n)/J_h$ is the polynomial part of $k[G_hT]$.[2]

Denote the character group of T by

$$\mathsf{X} := \mathbb{Z}^n \cong X(T).$$

For each $\lambda\in\mathsf{X}$, by [12, 3.1(13)(i)] (see also [3, 1.7] and [5]), there is a simple object $L_h(\lambda)$ in the category $\mathrm{Mod}(G_h)$ of G_h-modules and a simple object $\widehat{L}_h(\lambda)$ in the category $\mathrm{Mod}(G_hT)$ of G_hT-modules. Let

$$\mathsf{X}_h := X_h(T) = \{\lambda\in\mathsf{X}(=\mathbb{Z}^n) \mid 0\leqslant\lambda_i-\lambda_{i+1}\leqslant lp^{h-1}-1,\ 1\leqslant i\leqslant n\},$$

where we set $\lambda_{n+1}=0$. In particular, $\mathsf{X}_1 = \{\lambda\in\mathbb{Z}^n \mid 0\leqslant\lambda_i-\lambda_{i+1}<l, 1\leqslant i\leqslant n\}$,

Theorem 4.3. ([12, 3.1(13),(18)]) *The set $\{L_h(\lambda) \mid \lambda\in\mathsf{X}_h\}$ is a full set of nonisomorphic simple G_h-modules, and $\{\widehat{L}_h(\lambda) \mid \lambda\in\mathsf{X}\}$ is a full set of nonsiomorphic simple G_hT-modules. Moreover, for all $\lambda\in\mathsf{X}$, we have $\widehat{L}_h(\lambda)|_{G_h}\cong L_h(\lambda)$.*

By [3],[5] (cf. [10]), every polynomial G_hT-module (equivalently, $A_q(n)/J_h$-comodule) V has a direct sum decomposition $V=\oplus_{r\geqslant 0}V_r$, where V_r is the rth homogeneous component (i.e., is an $s_k(n,r)_h$-module). In particular, if $|\lambda|=r$, then $\widehat{L}_h(\lambda)$ is an $s_k(n,r)_h$-module. Note that a G_h-module does not have such a direct sum decomposition, since the decomposition $A_q(n)/I_h=\sum_{r\geqslant 0}\pi(A_q(n,r)_h)$ is not direct sum. However, if $|\lambda|=r$, then $L_h(\lambda)$ is a $u_k(n,r)_h$-module.

[2]If one introduces the quantum matrix semigroup M, its 'torus' D and the hth Frobenoius kernel M_h, then $A_q(n)$, $A_q(n)/I_h$, $A_q(n)/J_h$ are respectively the coordinate algebras of M, M_h and M_hD.

We now relate the q-Schur algebras to quantum enveloping algebra of $\mathfrak{gl}_n$ as given in [1], and define little q-Schur algebras.

Let $\Xi^{\pm}(n)$ be the set of all $A \in \Xi(n)$ whose diagonal entries are zero. Given $r > 0$, $A \in \Xi^{\pm}(n)$ and $\mathbf{j} = (j_1, j_2, \cdots, j_n) \in \mathbb{Z}^n$, we define

$$A(\mathbf{j}, r) = \sum_{\substack{D \in \Xi^0(n) \\ \sigma(A+D)=r}} v^{\sum_i d_i j_i}[A + D] \in \mathbf{U}(n, r) := U_{\mathbb{Q}(v)}(n, r).$$

where $\Xi^0(n)$ denotes the subset of diagonal matrices in $\Xi(n)$ and $D = \mathrm{diag}(d_1, \cdots, d_n)$.

The following result follows from [1, 5.5,5.7] (see also [20, (5.7)], [14, A.1] and [17, 3.4]). For $1 \leqslant i, j \leqslant n$, let $E_{i,j} \in \Xi(n)$ be the matrix unit $(a_{k,l})$ with $a_{k,l} = \delta_{i,k}\delta_{j,l}$.

Theorem 4.4. *There is an algebra epimorphism $\zeta_r : \mathbf{U}(n) \twoheadrightarrow \mathbf{U}(n, r)$ satisfying*

$$E_h \mapsto E_{h,h+1}(\mathbf{0}, r),\ K_1^{j_1} K_2^{j_2} \cdots K_n^{j_n} \mapsto 0(\mathbf{j}, r),\ F_h \mapsto E_{h+1,h}(\mathbf{0}, r).$$

Moreover, $\zeta_r(U_{\mathcal{Z}}(n)) = U_{\mathcal{Z}}(n, r)$ ([15]).

For $A \in \Xi(n)$ and for $\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{N}^n$, let $\left[{\mathtt{k}_i;c \atop t_i} \right] = \zeta_r\left(\left[{K_i;c \atop t_i} \right]\right)$, $\mathtt{k}_{\mathbf{t}} = \prod_{i=1}^{n} \left[{\mathtt{k}_i;0 \atop t_i} \right]$. Let

$$\mathtt{e}_i = \zeta_r(E_i), \quad \mathtt{f}_i = \zeta_r(F_i), \quad \mathtt{k}_j = \zeta_r(K_j) \quad \text{ for } 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n.$$

Let $U_{\mathcal{Z}}^+(n, r)$ (resp., $U_{\mathcal{Z}}^-(n, r)$, $U_{\mathcal{Z}}^0(n, r)$) be the $\mathcal{Z}$-subalgebras of $U_{\mathcal{Z}}(n, r)$ generated by the $\mathtt{e}_i^{(m)}$ (resp., $\mathtt{f}_i^{(m)}$, $\mathtt{k}_\lambda$), where $1 \leqslant i \leqslant n-1$ and $\lambda \in \Lambda(n, r) = \{\lambda \in \mathbb{N}^n \mid \sigma(\lambda) = r\}$. Here, $\sigma(\lambda) = \lambda_1 + \cdots + \lambda_n$.

Lemma 4.5. ([9, 19]) (1) *The set $\{\mathtt{k}_\lambda \mid \lambda \in \Lambda(n, r)\}$ is a complete set of primitive orthogonal idempotents (hence a basis) for $U_{\mathcal{Z}}^0(n, r)$. In particular, $1 = \sum_{\lambda \in \Lambda(n,r)} \mathtt{k}_\lambda$.*

(2) *Let $\lambda \in \Lambda(n, r)$, then $\mathtt{k}_i \mathtt{k}_\lambda = v^{\lambda_i} \mathtt{k}_\lambda$ for $1 \leqslant i \leqslant n$.*

Since $U_k(n, r) \cong U_{\mathcal{Z}}(n, r) \otimes_{\mathcal{Z}} k$, ζ_r naturally induces a surjective homomorphism $\zeta_r \otimes 1 : U_k(n) \twoheadrightarrow U_k(n, r)$. For convenience, we shall denote $\zeta_r \otimes 1$ by ζ_r. Similarly, we denote $\mathtt{e}_i \otimes 1$, $\mathtt{f}_i \otimes 1$, $\mathtt{k}_j \otimes 1$ by $\mathtt{e}_i$, $\mathtt{f}_i$, $\mathtt{k}_j$.

The algebra $\tilde{u}_k(n, r) := \zeta_r(\tilde{u}_k(n))$ is called a *little q-Schur algebra* in [18, 24] and is generated by $\mathtt{e}_i$, $\mathtt{f}_i$, $\mathtt{k}_j$. Putting $\tilde{u}_k^+(n, r) = \zeta_r(\tilde{u}_k^+(n))$, $\tilde{u}_k^-(n, r) = \zeta_r(\tilde{u}_k^-(n))$ and $\tilde{u}_k^0(n, r) = \zeta_r(\tilde{u}_k^0(n))$, we have $\tilde{u}_k(n, r) = \tilde{u}_k^-(n, r)\tilde{u}_k^0(n, r)\tilde{u}_k^+(n, r)$. Let

$$s_k(n, r) = \tilde{u}_k^-(n, r) U_k^0(n, r) \tilde{u}_k^+(n, r).$$

This is the subalgebra of $U_k(n, r)$ generated by the elements $\mathtt{e}_i$, $\mathtt{f}_i$, $\mathtt{k}_j$, $\left[{\mathtt{k}_j;0 \atop t} \right]$ ($1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$, $t \in \mathbb{N}$). We shall see below that $s_k(n, r)$ is isomorphic to the infinitesimal q-Schur algebra $s_k(n, r)_1$.

Remark 4.6. When $l' = l$ is odd, the restriction $\zeta_r : \tilde{u}_k(n) \to \tilde{u}_k(n, r)$ factors through the quotient algebra $u_k(n)$, the infinitesimal quantum $\mathfrak{gl}_n$, defined at the end of §2. Thus, in this case, $\tilde{u}_k(n, r)$ is the same algebra as $u_k(n, r)$ considered in [18].

For a positive integer m, let $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Let

$$\bar{(\ )}_m : \mathbb{Z}^n \to (\mathbb{Z}_m)^n$$

be the map defined by $\overline{(j_1, j_2, \cdots, j_n)} = (\overline{j_1}, \overline{j_2}, \cdots, \overline{j_n})$. For a subset Y of $\mathbb{Z}^n$, we shall denote $\overline{Y}_m = \{\bar{y} \in (\mathbb{Z}_m)^n \mid y \in Y\}$.

For $\overline{\lambda} \in (\mathbb{Z}_{l'})^n$, define

$$\mathtt{p}_{\overline{\lambda}} = \begin{cases} \sum\limits_{\mu\in\Lambda(n,r),\overline{\mu}=\overline{\lambda}} \mathtt{k}_\mu & \text{if } \overline{\lambda} \in \overline{\Lambda(n,r)}_{l'} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.7. [18, 24] *The set* $\{\mathtt{p}_{\overline{\lambda}} \mid \overline{\lambda} \in \overline{\Lambda(n,r)}_{l'}\}$ *forms a k-basis of* $\tilde{u}^0_k(n,r)$.

Let $\Xi(n)_h$ be the set of all $A = (a_{ij}) \in \Xi(n)$ such that $a_{ij} < lp^{h-1}$ for all $i \neq j$. Let $\Xi(n)^{\pm}_h = \{A \in \Xi(n)_h \mid a_{i,i} = 0, \forall i\}$. Let $\Xi'(n)_h$ be the set of all $n \times n$ matrices $A = (a_{ij})$ with $a_{ij} \in \mathbb{N}$, $a_{ij} < lp^{h-1}$ for all $i \neq j$ and $a_{ii} \in \mathbb{Z}_{l'p^{h-1}}$ for all i. We have an obvious map $pr : \Xi(n)_h \to \Xi'(n)_h$ defined by reducing the diagonal entries modulo $l'p^{h-1}$. We denote $\Xi(n,r)_h := \{A \in \Xi(n)_h \mid \sigma(A) = r\}$ and $\Xi(n,r)^{\pm}_h = \Xi(n,r)_h \cap \Xi(n)^{\pm}_h$.

Clearly, by regarding $s_k(n,r)_h$ as a subalgebra of the q-Schur algebra $U_k(n,r)$, the set

$$\{[A] \mid A \in \Xi(n,r)_h\}$$

forms a k-basis for $s_k(n,r)_h$.

Assume $A \in \Xi(n)^{\pm}_h$ with $\sigma(A) \leqslant r$. Given $\overline{\lambda} \in \overline{\Lambda(n, r-\sigma(A))}_{l'p^{h-1}}$, let

$$[\![A + \mathrm{diag}(\overline{\lambda}), r]\!]_h = \sum_{\substack{\mu\in\Lambda(n,r-\sigma(A))\\ \overline{\mu}=\overline{\lambda}}} [A + \mathrm{diag}(\mu)]. \tag{4.7.1}$$

Lemma 4.8. *The set*

$$\{[\![A + \mathrm{diag}(\overline{\lambda}), r]\!]_h \mid A \in \Xi(n,r)^{\pm}_h, \overline{\lambda} \in \overline{\Lambda(n, r-\sigma(A))}_{l'p^{h-1}}\}$$

forms a k-basis for $\tilde{u}_k(n,r)_h$. Thus $\dim_k \tilde{u}_k(n,r)_h = \# pr(\Xi(n,r)_h)$. *Similarly, the set*

$$\Big\{ \sum_{\substack{\mu\in\Lambda(n,r-\sigma(A))\\ \overline{\mu}=\overline{\lambda}}} \xi_{A+\mathrm{diag}(\mu)} \mid A \in \Xi(n,r)^{\pm}_h, \overline{\lambda} \in \overline{\Lambda(n, r-\sigma(A))}_{lp^{h-1}} \Big\}$$

forms a k-basis for $u_k(n,r)_h$.

Proof. By [23, 4.2.4], the set

$$\{c^{A+\mathrm{diag}(\overline{\lambda})} + \tilde{I}_h \mid A \in \Xi(n,r)^{\pm}_h, \overline{\lambda} \in \overline{\Lambda(n, r-\sigma(A))}_{l'p^{h-1}}\} \tag{4.8.1}$$

forms a k-basis for $\tilde{\pi}(A_q(n,r)_h)$. Similar to [23, 5.5.3],[3] we have

$$(c^{A+\mathrm{diag}(\overline{\lambda})} + \tilde{I}_h)^* = \sum_{\substack{\mu\in\Lambda(n,r-\sigma(A))\\ \overline{\mu}=\overline{\lambda}}} \xi_{A+\mathrm{diag}(\mu)} = \varepsilon^{d_{A+\mathrm{diag}(\lambda)}} [\![A + \mathrm{diag}(\overline{\lambda}), r]\!]_h. \tag{4.8.2}$$

[3]The argument given in [23] is for the quantum coordinate algebra $A_{q,1}(n)$, while $A_q(n)$ considered here is $A_{1,q}(n)$. Here $A_{\alpha,\beta}(n)$ is the two parameter version defined in [39].

Here baring on μ is relative to $l'p^{h-1}$. The first assertion follows. Replacing $l'p^{h-1}$ by lp^{h-1} in the bar map, the first quality of (4.8.2) gives the second assertion. □

The proof above gives immediately the following result. This result, in terms of two parameter quantum linear groups, is the $(1,q)$-version of [23, 5.5] which is the $(q,1)$-version, see the footnote above.

Corollary 4.9. *We have algebra isomorphisms $\tilde{u}_k(n,r)_1 \cong \tilde{u}_k(n,r)$ and $s_k(n,r)_1 \cong s_k(n,r)$.*

Note that $u_k(n,r)$ (see 4.6) is not defined when l' is even. However, $u_k(n,r)_1$ is always defined, regardless l' is odd or even.

5. The classification of simple modules of little q-Schur algebras

In this section, we shall give the classification of simple modules for the little q-Schur algebra $\tilde{u}_k(n,r)$.

For $\overline{\lambda} \in (\mathbb{Z}_{l'})^n$, let $\mathfrak{M}_k(\overline{\lambda}) = \tilde{u}_k(n)/\mathfrak{I}_k(\overline{\lambda})$ where

$$\mathfrak{I}_k(\overline{\lambda}) = \sum_{1\leqslant i\leqslant n-1} \tilde{u}_k(n)E_i + \sum_{1\leqslant i\leqslant n} \tilde{u}_k(n)(K_i - \varepsilon^{\lambda_i}).$$

Then $\mathfrak{M}_k(\overline{\lambda})$ has a unique irreducible quotient, which will be denoted by $\mathfrak{L}_k(\overline{\lambda})$(see the proof of [35, 5.11]). If l' is odd, then $K_i^{l'}$ is central in $\tilde{u}_k(n)$ and hence $\mathfrak{L}_k(\overline{\lambda})$ can be regarded as a $u_k(n)$-module.

Let $\Lambda^+(n,r) = \{\lambda \in \Lambda(n,r) \mid \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n\}$ and let $\Lambda^+(n) = \cup_{r\geqslant 0}\Lambda^+(n,r)$. For $\lambda \in \Lambda^+(n,r)$ let $V(\lambda)$ be the simple $\mathbf{U}(n,r)$-module with highest weight λ. Let x_λ be the highest weight vector of $V(\lambda)$. Let $V_{\mathcal{Z}}(\lambda) = U_{\mathcal{Z}}(n,r)x_\lambda$. Since $U_{\mathcal{Z}}(n,r)$ is a homomorphic image $U_{\mathcal{Z}}(\mathfrak{sl}_\mathfrak{n})$ by 9.3, we have $V_{\mathcal{Z}}(\lambda) = U_{\mathcal{Z}}(\mathfrak{sl}_\mathfrak{n})x_\lambda$. We denote $V_k(\lambda) = V_{\mathcal{Z}}(\lambda) \otimes_{\mathcal{Z}} k$ and let $L_k(\lambda)$ be the unique irreducible quotient of $V_k(\lambda)$. For convenience, we shall denote the image of x_λ in $V_k(\lambda)$ and $L_k(\lambda)$ by the same letter. Let $V_k'(\lambda) = \tilde{u}_k(n)x_\lambda$. We call $V_k'(\lambda)$ the baby Weyl module of $\tilde{u}_k(n)$. Then $\mathfrak{L}_k(\overline{\lambda})$ is the unique irreducible quotient of $V_k'(\lambda)$.

If $\lambda \in \mathsf{X}_1$, then, by 3.2, and 3.7, $\tilde{u}_k(\mathfrak{sl}_n)x_\lambda = V_k(\lambda)$ and $L_k(\lambda)|_{\tilde{u}_k(\mathfrak{sl}_n)}$ is irreducible. This together with $\tilde{u}_k(\mathfrak{sl}_n) \subseteq \tilde{u}_k(\mathfrak{gl}_n) \subseteq U_k(n)$ implies the following.

Lemma 5.1. *For any $\lambda \in \mathsf{X}_1$, we have $V_k'(\lambda) = V_k(\lambda)$ and restriction gives $\tilde{u}_k(n)$-module isomorphisms $L_k(\lambda)|_{\tilde{u}_k(n)} \cong \mathfrak{L}_k(\overline{\lambda})$.*

Note further that $\overline{(\mathsf{X}_1)}_{l'} \subseteq \overline{\Lambda^+(n)}_{l'} = (\mathbb{Z}_{l'})^n$ and

$$\overline{(\mathsf{X}_1)}_{l'} = \overline{\Lambda^+(n)}_{l'} = (\mathbb{Z}_{l'})^n, \quad \text{if } l' \text{ is odd.} \tag{5.1.1}$$

Theorem 5.2 ([35, 5.11,6.6]). *(1) If l' is odd, then $l' = l$ and the set $\{\mathfrak{L}_k(\nu) \mid \nu \in (\mathbb{Z}_l)^n\}$ forms a complete set of non-isomorphic simple $u_k(n)$-modules.*

(2) If l' is even, then $l' = 2l$ and the set $\{\mathfrak{L}_k(\nu) \mid \nu \in (\mathbb{Z}_{2l})^n\}$ forms a complete set of non-isomorphic simple $\tilde{u}_k(n)$-modules.

Note that, unlike the classification for simple G_1-modules given in Theorem 4.3, this classification depends on l'. We will make a comparison in Corollary 5.7.

By 5.1 and (5.1.1), if l' is odd, then every simple $\tilde{u}_k(n)$-module on which all K_i^l act as the identity is a $u_k(n)$-module and is also a restriction of a simple $U_k(n)$-module with a restricted highest weight. However, when l' is even, there are simple $\tilde{u}_k(n)$-modules which cannot be realized in this way.

Example 5.3. Assume $l' = 4$. Then $l = 2$. Let $V(2,0) = \mathbf{U}(2)/I(2,0)$, where

$$I(2,0) = \mathbf{U}(2))E_1 + \mathbf{U}(2)F_1^3 + \mathbf{U}(2))(K_1 - v^2)) + \mathbf{U}(2))(K_2 - 1)).$$

Let $x_0 = 1 + I(2,0) \in V(2,0)$. Let $V_{\mathcal{Z}}(2,0)$ be the $U_{\mathcal{Z}}(2)$-submodule of $V(2,0)$ generated by x_0. Let $V_k(2,0) = V_{\mathcal{Z}}(2,0) \otimes_{\mathcal{Z}} k$. Let $V_k'(2,0) = \tilde{u}_k(2)x_0$. The set $\{x_0, F_1x_0, F_1^{(2)}x_0\}$ forms a k-basis for $V_k(2,0)$. Since $F_1^2 = 0 \in U_k(2)$, $V_k'(2,0) = \mathrm{span}_k\{x_0, F_1x_0\}$. Thus $V_k'(2,0) \neq V_k(2,0)$. Since $E_1(F_1x_0) = 0$, $\mathrm{span}_k\{F_1x_0\}$ is a submodule of $V_k'(2,0)$ (resp., $V_k(2,0)$). Hence, $L_k'(2,0)$ is one dimensional, while $\dim L_k(2,0) = 2$.

By regarding $L_k'(2,0)$ as the $\tilde{u}_k(\mathfrak{sl}_2)$-module $L_k'(2)$, we see that there is no simple $U_k(\mathfrak{sl}_2)$-module $L_k(m)$ such that $L_k(m)|_{\tilde{u}_k(\mathfrak{sl}_2)} \cong L_k'(2)$.

We are now ready to classify simple $\tilde{u}_k(n,r)$-modules.

Lemma 5.4. *Let L be a $\tilde{u}_k(n,r)$-module. Assume $x_0 \neq 0 \in L$ satisfies $k_i x_0 = \varepsilon^{\lambda_i} x_0$ for some $\lambda_i \in \mathbb{N}$ with $1 \leqslant i \leqslant n$. Then $\overline{\lambda} = (\overline{\lambda}_1, \cdots, \overline{\lambda}_n) \in \overline{\Lambda(n,r)}_{l'}$.*

Proof. By 4.5 and 4.7, we have $1 = \sum_{\alpha \in \overline{\Lambda(n,r)}_{l'}} \mathtt{p}_\alpha$ and $\mathtt{p}_\alpha \in \tilde{u}_k(n,r)$. It follows that $x_0 = \sum_{\alpha \in \overline{\Lambda(n,r)}_{l'}} (\mathtt{p}_\alpha x_0)$ and hence, there exist $\beta \in \overline{\Lambda(n,r)}_{l'}$ such that $\mathtt{p}_\beta x_0 \neq 0$. By 4.5,

$$\varepsilon^{\lambda_i} x_0 = \mathtt{k}_i x_0 = \mathtt{k}_i \sum_{\alpha \in \overline{\Lambda(n,r)}_{l'}} \mathtt{p}_\alpha x_0 = \sum_{\alpha \in \overline{\Lambda(n,r)}_{l'}} \varepsilon^{\alpha_i} \mathtt{p}_\alpha x_0$$

for $1 \leqslant i \leqslant n$. Hence,

$$\varepsilon^{\lambda_i} \mathtt{p}_\beta x_0 = \mathtt{p}_\beta(\varepsilon^{\lambda_i} x_0) = \mathtt{p}_\beta \sum_{\alpha \in \overline{\Lambda(n,r)}_{l'}} \varepsilon^{\alpha_i} \mathtt{p}_\alpha x_0 = \varepsilon^{\beta_i} \mathtt{p}_\beta x_0$$

for $1 \leqslant i \leqslant n$. Since $\mathtt{p}_\beta x_0 \neq 0$, we have $\varepsilon^{\lambda_i} = \varepsilon^{\beta_i}$ for $1 \leqslant i \leqslant n$ and hence, $\overline{\lambda} = \beta \in \overline{\Lambda(n,r)}_{l'}$. $\square$

Let

$$\mathsf{X}_h(l) = \mathsf{X}_h + l\mathbb{N}^n \quad \text{and} \quad \mathsf{X}_h(l,r) = \{\lambda \in \mathsf{X}_h(l) \mid \sigma(\lambda) = r\}.$$

For $h = 1$ and $\lambda \in \mathsf{X}_1(l,r)$, the irreducible (polynomial) G_1T-module $\widehat{L}_1(\lambda)$ given in 4.3 is in fact an irreducible $A_q(n,r)_1$-comodule. Hence, $\widehat{L}_1(\lambda)$ has a natural $s_k(n,r)_1$-module structure.

Theorem 5.5. *For $\lambda \in \mathsf{X}_1(l,r)$ we have $\widehat{L}_1(\lambda)|_{\tilde{u}_k(n,r)} \cong \mathfrak{L}_k(\overline{\lambda})$. Moreover the set $\{\mathfrak{L}_k(\overline{\lambda}) \mid \overline{\lambda} \in \overline{\mathsf{X}_1(l,r)}_{l'}\}$ forms a complete set of non-isomorphic simple $\tilde{u}_k(n,r)$-modules.*

Proof. By [3, 5] (cf. 4.3),

$$\text{the set } \{\widehat{L}_1(\lambda) \mid \lambda \in \mathsf{X}_1(l,r)\} \text{ forms a complete set of non-isomorphic simple } s_k(n,r)_1\text{-modules.} \tag{5.5.1}$$

Thus, it is enough to prove that for each $\lambda \in \mathsf{X}_1(l,r)$, $\widehat{L}_1(\lambda)|_{\tilde{u}_k(n,r)}$ is irreducible, and every irreducible $\tilde{u}_k(n,r)$-module is isomorphic to $\mathfrak{L}_k(\overline{\mu})$ for some $\overline{\mu} \in \overline{\mathsf{X}_1(l,r)}_{l'}$

By 4.3, for $\lambda \in \mathsf{X}_1(l,r)$, we see that $\widehat{L}_1(\lambda)|_{G_1}$ is a simple G_1-module at level r. Hence, by (4.2.1), $\widehat{L}_1(\lambda)|_{u_k(n,r)_1}$ is a simple $u_k(n,r)_1$-module. Now the inclusions $u_k(n,r)_1 \subseteq \tilde{u}_k(n,r) \subseteq s_k(n,r)_1 = s_k(n,r)$ given in (4.1.2) force that $\widehat{L}_1(\lambda)|_{\tilde{u}_k(n,r)}$ is a simple $\tilde{u}_k(n,r)$-module. Hence, inflation gives a simple $\tilde{u}_k(n)$-module. Since $\widehat{L}_1(\lambda)$ is a highest weight $s_k(n,r)$-module, by [35, 5.10(b)], there exists $x_0 \in \widehat{L}_1(\lambda)$ such that $E_i x_0 = 0$ and $K_i x_0 = \varepsilon^{\lambda_i} x_0$ for all i. Now, the argument in [35, 5.11] implies that $\widehat{L}_1(\lambda)|_{\tilde{u}_k(n,r)}$ is isomorphic to $\mathfrak{L}_k(\overline{\lambda})$.

On the other hand, let L be a simple $\tilde{u}_k(n,r)$-module, then, by inflation, L is a simple $\tilde{u}_k(n)$-module. (If l' is an odd number, L is also a simple $u_k(n)$-module.) Hence, there is some $x_0 \neq 0 \in L$ such that $E_i x_0 = 0$ and $k_j x_0 = \varepsilon^{\lambda_j} x_0$ for $1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n$, where $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{N}^n$. By 5.4, we have $\bar{\lambda} \in \overline{\Lambda(n,r)}_{l'}$. So, without loss, we may choose $\lambda \in \Lambda(n,r)$. We consider the $s_k(n,r)$-module $s_k(n,r) \otimes_{\tilde{u}_k(n,r)} L$. Let V be the $s_k(n,r)$-submodule of $s_k(n,r) \otimes_{\tilde{u}_k(n,r)} L$ generated by $\mathtt{k}_\lambda \otimes x_0$. Then $V = s_k(n,r)(\mathtt{k}_\lambda \otimes x_0) = \tilde{u}_k^-(n,r)(\mathtt{k}_\lambda \otimes x_0)$. It is clear that the $\mathtt{k}_\lambda \otimes x_0$ is the highest weight vector of V. Hence there is a unique maximal $s_k(n,r)$-submodule of V, say V_{max}. Let $L' = V/V_{max}$. Then L' is a simple $s_k(n,r)$-module and hence $L' \cong \widehat{L}_1(\lambda)$ by (5.5.1). Since $\mathtt{e}_i.(\mathtt{k}_\lambda \otimes x_0) = 0$ and $\mathtt{k}_j.(\mathtt{k}_\lambda \otimes x_0) = \varepsilon^{\lambda_j}(\mathtt{k}_\lambda \otimes x_0)$, by 5.2, we have $L \cong \widehat{L}_1(\lambda)|_{\tilde{u}_k(n,r)} \cong \mathfrak{L}_k(\overline{\lambda})$. The proof is completed. □

Corollary 5.6. *Every simple $u_k(n)$-module when $l' = l$ is odd (resp. every simple $\tilde{u}_k(n)$-module when l' is even) is an inflation of a simple $\tilde{u}_k(n,r)$-module for some r.*

Proof. Since $\mathbb{Z}^n = \mathsf{X}_1(T) + l\mathbb{Z}^n$, it follows that $\bigcup_{r \geqslant 0} \overline{\mathsf{X}_1(l,r)}_{l'} = \overline{\mathsf{X}_1(l)}_{l'} = \overline{(\mathsf{X}_1)}_{l'} + \overline{(l\mathbb{N}^n)}_{l'} = \overline{(\mathsf{X}_1)}_{l'} + \overline{(l\mathbb{Z}^n)}_{l'} = \mathbb{Z}^n_{l'}$. Thus, by 5.5, inflation via epimorphisms $\tilde{u}_k(n) \to \tilde{u}_k(n,r)$ gives simple $\tilde{u}_k(n)$-modules indexed by $\mathbb{Z}^n_{l'}$. Now the assertion follows from 5.2. □

We remark that restricted simple $U_k(n)$-modules $L_k(\lambda)$ with $\lambda \in \mathsf{X}_1$ does not cover all simple $\tilde{u}_k(n)$-module when l' is even. The above result shows that simple G_1T-modules *does* cover all simple $\tilde{u}_k(n)$-module.

Corollary 5.7. *We have, for $\lambda \in \mathsf{X}_1$ with $\sigma(\lambda) = r$, $L_1(\lambda) \cong \mathfrak{L}_k(\overline{\lambda})|_{u_k(n,r)_1}$ where $\overline{\lambda} \in \overline{(\mathsf{X}_1)}_{l'}$. In other words, $\{\mathfrak{L}_k(\nu) \mid \nu \in \overline{(\mathsf{X}_1)}_{l'}\}$ is a complete set of all simple G_1-modules.*

Note that this classification is the same as the one given in 4.3 since the set $\overline{(\mathsf{X}_1)}_{l'}$ can be identified with X_1 via the map $\mathsf{X}_1 \to \overline{(\mathsf{X}_1)}_{l'}, \lambda \mapsto \overline{\lambda}$. Thus, for the example $L'_k(2,0)$ constructed in 5.3, its restriction to G_1 is again irreducible and isomorphic to $L_1(0,0)$.

Remarks 5.8. (1) If $\lambda \in \mathsf{X}_1$, 5.1 and 5.5 imply that restriction induces isomorphisms $L_k(\lambda) \cong \widehat{L}_1(\lambda)$, $\widehat{L}_1(\lambda) \cong \mathfrak{L}_k(\overline{\lambda})$ and $\mathfrak{L}_k(\overline{\lambda}) \cong L_1(\lambda)$.

(2) When $l' = l$ is odd, we established in [18] that a basis for $\tilde{u}_k(n,r)$ is indexed by $\overline{\Xi(n,r)}_l$, while a basis for $U_k(n,r)$ is indexed by $\Xi(n,r)$. Similarly, since $\overline{\Lambda^+(n,r)}_l = \overline{\mathsf{X}_1(l,r)}_l$, simple $\tilde{u}_k(n,r)$-modules are indexed by $\overline{\Lambda^+(n,r)}_l$, while simple $U_k(n,r)$-modules are indexed by $\Lambda^+(n,r)$. Thus, baring on the index sets gives the counterparts for little q-Schur algebras. However, if l' is even, then $\overline{\Lambda^+(n,r)}_{l'}$ is even *not* a subset of $\overline{\mathsf{X}_1(l,r)}_{l'}$ and the classification is quite different.

Note that, as a comparison, (5.5.1) shows that the classification for the infinitesimal q-Schur algebra is independent of l'.

6. The baby transfer map

There is an epimorphism $\psi_{r+n,r} : U_k(n,r+n) \twoheadrightarrow U_k(n,r)$, called the transfer map in [36, §2]. This map can be geometrically constructed ([29] and [36, §2]) and algebraically constructed by quantum coordinate algebras and quantum determinant ([16, 5.4]). Since $\psi_{r+n,r}$ satisfies

$$\zeta_{r+n}(E_i) \longmapsto \zeta_r(E_i), \quad \zeta_{r+n}(F_i) \longmapsto \zeta_r(F_i), \quad \zeta_{r+n}(K_i) \longmapsto \varepsilon\zeta_r(K_i),$$

its restriction induces an epimorphism

$$\psi_{r+n,r} : \tilde{u}_k(n,r+n) \twoheadrightarrow \tilde{u}_k(n,r). \tag{6.0.1}$$

In this section, we introduce the *baby transfer map* $\rho_{r+l',r} : \tilde{u}_k(n,r+l') \to \tilde{u}_k(n,r)$ and use it to prove that, up to isomorphism, there only finitely many little q-Schur algebras. By these maps, we will understand the classification of simple modules for the little q-Schur algebras from a different angle.

Proposition 6.1. *There is an algebra epimorphism $\rho_{r+l',r} : \tilde{u}_k(n,r+l') \twoheadrightarrow \tilde{u}_k(n,r)$ satisfying*

$$\mathtt{e}'_i \mapsto \mathtt{e}_i, \quad \mathtt{f}'_i \mapsto \mathtt{f}_i, \quad \mathtt{k}'_j \mapsto \mathtt{k}_j,$$

where $\mathtt{e}'_i, \mathtt{f}'_i, \mathtt{k}'_i$ are the corresponding $\mathtt{e}_i, \mathtt{f}_i, \mathtt{k}_i$ for $\tilde{u}_k(n,r+l')$. Moreover, for $A \in \Xi^\pm(n)_1$ and $\overline{\lambda} \in \overline{\Lambda(n,r+l'-\sigma(A))}_{l'}$,

$$\rho_{r+l',r}([\![A+\mathrm{diag}(\overline{\lambda}), r+l']\!]) = \begin{cases} [\![A+\mathrm{diag}(\overline{\lambda}), r]\!] & \textit{if } \overline{\lambda} \in \overline{\Lambda(n,r-\sigma(A))}_{l'} \\ 0 & \textit{otherwise.} \end{cases} \tag{6.1.1}$$

Proof. Consider the epimorphism $\tilde{\pi} : A_q(n)/J_1 \to A_q(n)/\tilde{I}_1$ given in (4.1.1). Since every monomial m in $A_q(n,r)_1$ has the same homomorphic image as the monomial $c_{11}^{l'}m \in A_q(n,r+l')_1$, it follows that $\tilde{\pi}(A_q(n,r)_1) \subseteq \tilde{\pi}(A_q(n,r+l')_1)$ and the basis $\{c^{A+\mathrm{diag}(\lambda)} + \tilde{I}_1 \mid A \in \Xi^\pm(n)_1, \overline{\lambda} \in \overline{\Lambda(n,r-\sigma(A))}_{l'}\}$ for $\tilde{\pi}(A_q(n,r)_1)$ given in (4.8.1) extends to a basis for $\tilde{\pi}(A_q(n,r+l')_1)$. By taking dual and 4.9, there is an algebra epimorphism $\rho_{r+l',r} : \tilde{u}_k(n,r+l') \to \tilde{u}_k(n,r)$.

Let $\big\{(c^{A+\mathrm{diag}(\lambda)}+\tilde{I}_1)^* \mid A\in\Xi^\pm(n)_1,\, \overline{\lambda}\in\overline{\Lambda(n,r'-\sigma(A))}_{l'}\big\}$ be the dual basis for $\tilde{u}_k(n,r')$. It is now clear that, for $A\in\Xi^\pm(n)_1$ and $\overline{\lambda}\in\overline{\Lambda(n,r+l'-\sigma(A))}_{l'}$,

$$\rho_{r+l',r}((c^{A+\mathrm{diag}(\overline{\lambda})}+\tilde{I}_1)^*)=\begin{cases}(c^{A+\mathrm{diag}(\overline{\lambda})}+\tilde{I}_1)^* & \text{if } \overline{\lambda}\in\overline{\Lambda(n,r-\sigma(A))}_{l'}\\ 0 & \text{otherwise.}\end{cases}$$

This together with (4.8.2) implies (6.1.1) since ε is a primitive l'-th root of unity. Since $\mathtt{p}_{\overline{\lambda}}=[\![\mathrm{diag}(\overline{\lambda}),r]\!]$, we have for $\overline{\lambda}\in\overline{\Lambda(n,r+l')}_{l'}$

$$\rho_{r+l',r}(\mathtt{p}_{\overline{\lambda}})=\begin{cases}\mathtt{p}_{\overline{\lambda}} & \text{if } \overline{\lambda}\in\overline{\Lambda(n,r)}_{l'}\\ 0 & \text{otherwise.}\end{cases}$$

Hence, $\rho_{r+l',r}(\mathtt{k}_i')=\rho_{r+l',r}(\sum_{\overline{\lambda}\in\overline{\Lambda(n,r+l')}_{l'}}\varepsilon^{\lambda_i}\mathtt{p}_{\overline{\lambda}})=\sum_{\overline{\lambda}\in\overline{\Lambda(n,r)}_{l'}}\varepsilon^{\lambda_i}\mathtt{p}_{\overline{\lambda}}=\mathtt{k}_i$. Similarly, we can prove $\rho_{r+l',r}(\mathtt{e}_i')=\mathtt{e}_i$ and $\rho_{r+l',r}(\mathtt{f}_i')=\mathtt{f}_i$. □

Observe that, if $r\geqslant(n-1)(l'-1)$, then, for $\lambda\in\Lambda(n,r+l')$, $\sum_{1\leqslant i\leqslant n}\lambda_i=r+l'\geqslant n(l'-1)+1$. Thus, $\lambda_i\geqslant l'$ for some i. Consequently, $\overline{\lambda}=\overline{(\lambda_1,\cdots,\lambda_i-l',\cdots,\lambda_n)}\in\overline{\Lambda(n,r)}_{l'}$. Thus, we see that

$$\overline{\Lambda(n,r)}_{l'}=\overline{\Lambda(n,r+l')}_{l'} \text{ whenever } r\geqslant(n-1)(l'-1). \tag{6.1.2}$$

Corollary 6.2. *We have $\widetilde{u}_k(n,r)\cong\widetilde{u}_k(n,r+l')$ for $r\geqslant(l-1)(n^2-n)+(n-1)(l'-1)$. Hence, up to isomorphism, there are only finitely many little q-Schur algebras.*

Proof. By 6.1, there exists an algebra epimorphism from $\widetilde{u}_k(n,r+l')$ to $\widetilde{u}_k(n,r)$. Thus, it is enough to prove that $\dim_k\widetilde{u}_k(n,r+l')=\dim_k\widetilde{u}_k(n,r)$ for $r\geqslant(l-1)(n^2-n)+(n-1)(l'-1)$. By [18, 8.2] and [24, 6.8], we have

$$\dim_k\tilde{u}_k(n,r)=|\{(A,\overline{\lambda})\mid A\in\Xi^\pm(n)_1,\ \overline{\lambda}\in\overline{\Lambda(n,r-\sigma(A))}_{l'}\}|. \tag{6.2.1}$$

If $r\geqslant(l-1)(n^2-n)+(n-1)(l'-1)$, then, for any $A\in\Xi^\pm(n)_1$,

$$r-\sigma(A)\geqslant r-(n^2-n)(l-1)\geqslant(n-1)(l'-1).$$

Thus, by (6.1.2), $\overline{\Lambda(n,r-\sigma(A))}_{l'}=\overline{\Lambda(n,r-\sigma(A)+l')}_{l'}$. Consequently, (6.2.1), implies $\dim_k\widetilde{u}_k(n,r+l')=\dim_k\widetilde{u}_k(n,r)$ whenever $r\geqslant(l-1)(n^2-n)+(n-1)(l'-1)$. This completes the proof. □

We now look at the second application of the baby transfer map. If l' is odd, then $l'=l$ by definition and the index set of the classification given in 5.5 becomes

$$\begin{aligned}\overline{\mathsf{X}_1(l,r)}_l&=\{\overline{\lambda}\mid\lambda\in\mathsf{X}_1,\sigma(\lambda)\leqslant r,\overline{\sigma(\lambda)}=\overline{r}\}\\&=\overline{\mathsf{X}_1(l,r-l)}_l\cup\{\overline{\lambda}\mid\lambda\in\mathsf{X}_1,\ \sigma(\lambda)=r\}.\end{aligned} \tag{6.2.2}$$

This indicates, by 5.5 and 6.1, that the simple $\tilde{u}_k(n,r)$-modules can be divided into two classes, one consists of the simple $\tilde{u}_k(n,r)$-modules which can be obtained by restriction from the simple

$U_k(n,r)$-modules with restricted highest weights and the other consists of the simple $\tilde{u}_k(n,r)$-module which are inflations of the simple $\tilde{u}_k(n,r-l)$-module via the map $\rho_{r,r-l}$. The disjointness of the two classes can be seen as follows.

Suppose $n \geqslant r$. Let $\omega = (1^r) \in \Lambda(n,r)$. Then $\mathtt{k}_\omega = \mathtt{p}_\omega \in \tilde{u}_k(n,r)$. By [23, 7.1] we have $\mathtt{k}_\omega \tilde{u}_k(n,r)\mathtt{k}_\omega$ is isomorphic to the Hecke algebra $\mathcal{H}_r = \mathcal{H}(\mathfrak{S}_r)$. We will identify $\mathtt{k}_\omega \tilde{u}_k(n,r)\mathtt{k}_\omega$ with $\mathcal{H}_r$. Thus, we may define the "baby" Schur functor F_r as follows:

$$F_r : \mathrm{Mod}(\tilde{u}_k(n,r)) \longrightarrow \mathrm{Mod}(\mathcal{H}_r), \quad V \longmapsto \mathtt{k}_\omega V.$$

The functor F_r induces a group homomorphism over the Grothendieck groups

$$\mathtt{F}_r : K(\tilde{u}_k(n,r)) \to K(\mathcal{H}_r).$$

Here $K(A)$ denotes the Grothendieck group of $\mathrm{Mod}(A)$.

By 6.1 the category $\mathrm{Mod}(\tilde{u}_k(n,r-l'))$ can be regarded as a full subcategory of $\mathrm{Mod}(\tilde{u}_k(n,r))$ via $\rho_{r+l',r}$ and hence we may view $K(\tilde{u}_k(n,r-l'))$ as a subgroup of $K(\tilde{u}_k(n,r))$.

Proposition 6.3. *Assume l' is odd and $n \geqslant r$. Then $\mathtt{F}_r$ is surjective and $\ker(\mathtt{F}_r) = K(\tilde{u}_k(n,r-l))$.*

Proof. By 5.1 and [12, 4.4(2)], the set $\{F_r(\mathfrak{L}_k(\overline{\lambda})) \mid \lambda \in \mathsf{X}_1,\, \sigma(\lambda) = r\}$ forms a complete set of non-isomorphic simple $\mathcal{H}_r$-modules. Thus by [28, (6.2(g))] and (6.2.2), we conclude that $F_r(\mathfrak{L}_k(\overline{\lambda})) = 0$ for $\overline{\lambda} \in \overline{\mathsf{X}_1(l,r-l)}_l$. The assertion follows. □

7. Semisimple little q-Schur algebras

We now determine semisimple little q-Schur algebras. This can be easily done by the semisimplicity of the infinitesimal q-Schur algebra $s_k(n,r) = s_k(n,r)_1$ and the following.

Lemma 7.1. *Let V be an $s_k(n,r)$-module. Then $\mathrm{soc}_{s_k(n,r)} V = \mathrm{soc}_{\tilde{u}_k(n,r)} V$.*

Proof. It is easy to check that $\mathrm{soc}_{s_k(n,r)_1} V = \mathrm{soc}_{G_1T} V$ and $\mathrm{soc}_{u_k(n,r)_1} V = \mathrm{soc}_{G_1} V$. By [12, 3.1(18)(iii)] we have $\mathrm{soc}_{G_1T} V = \mathrm{soc}_{G_1} V$. It follows that $\mathrm{soc}_{s_k(n,r)_1} V = \mathrm{soc}_{u_k(n,r)_1} V$. Since $u_k(n,r)_1 \subseteq \tilde{u}_k(n,r) \subseteq s_k(n,r)_1$ the assertion follows from 4.3 and 5.5. □

Theorem 7.2. *The little q-Schur algebra $\tilde{u}_k(n,r)$ is semisimple if and only if either $l > r$ or $l = n = 2$ and $r \geqslant 3$ is odd.*

Proof. By [26, 1.2], the infinitesimal q-Schur algebra $s_k(n,r)$ is semisimple if and only if either $l > r$ or $n = 2$, $l = 2$ and $r \geqslant 3$ is odd. Thus, it is enough to prove that the infinitesimal q-Schur algebra $s_k(n,r)$ is semisimple if and only if the little q-Schur algebra $\tilde{u}_k(n,r)$ is semisimple.

Suppose the algebra $s_k(n,r)$ is semisimple. Let W be an indecomposable projective $\tilde{u}_k(n,r)$-module. Since $s_k(n,r)$ is semisimple, the $s_k(n,r)$-module $s_k(n,r) \otimes_{\tilde{u}_k(n,r)} W$ is semisimple. By 5.5, $(s_k(n,r) \otimes_{\tilde{u}_k(n,r)} W)|_{\tilde{u}_k(n,r)}$ is a semisimple $\tilde{u}_k(n,r)$-module. Since W is a projective $\tilde{u}_k(n,r)$-module, W is a flat $\tilde{u}_k(n,r)$-module. It follows that the natural $\tilde{u}_k(n,r)$-module homomorphism

from $W \cong \tilde{u}_k(n,r) \otimes_{\tilde{u}_k(n,r)} W$ to $s_k(n,r) \otimes_{\tilde{u}_k(n,r)} W$ is injective and hence W is a semisimple $\tilde{u}_k(n,r)$-module. So the algebra $\tilde{u}_k(n,r)$ is semisimple.

Now we suppose the algebra $s_k(n,r)$ is not semisimple. Then there exist $\lambda, \mu \in \mathsf{X}_1(l,r)$ such that $\mathrm{Ext}_{s_k(n,r)}(\widehat{L}_1(\lambda), \widehat{L}_1(\mu)) \neq 0$ and hence, there exists an $s_k(n,r)$-module V such that $\mathrm{soc}_{s_k(n,r)} V = \widehat{L}_1(\mu)$ with top $\widehat{L}_1(\lambda)$. By 7.1 we have $\mathrm{soc}_{\tilde{u}_k(n,r)} V = \widehat{L}_1(\mu)$ and hence $\tilde{u}_k(n,r)$ is not semisimple. □

We will see in the next section (at least when l' is odd) that the semisimplicity of $\tilde{u}_k(n,r)$ depends only on r and l, while the infinitesimal quantum group $\tilde{u}_k(n)$ is never semisimple (for all n and $l' = l$).

8. Little q-Schur algebras of finite representation type

In this section, we will assume k is an algebraically closed field and l' is odd. Thus, $l' = l \geqslant 3$ and $\overline{\mathsf{X}_1(l,r)}_l = \overline{\Lambda^+(n,r)}_l$ (see 5.8(2)). By 4.3 and 5.5, $L_1(\lambda) = \mathfrak{L}_k(\overline{\lambda})$ for all $\lambda \in \mathsf{X}_1(l,r)$. We will classify little q-Schur algebras of finite representation type in this case. The even case is much more complicated and will be treated elsewhere.

We first determine the blocks of $\tilde{u}_k(2,r)$. Using it, we then establish that $\tilde{u}_k(2,r)$ has finite representation type if and only if it is semisimple. We then generalize this from $n = 2$ to an arbitrary n.

Blocks of q-Schur algebras were classified in [3, 4] (cf. also [11]). Moreover, blocks of infinitesimal q-Schur algebras were classified in [3, 5] for $n = 2$. Now we first classify blocks of little q-Schur algebras $\tilde{u}_k(2,r)$ and use this to determine their finite representation type.

Let $\Phi = \{\boldsymbol{e}_i - \boldsymbol{e}_j \mid 1 \leqslant i \neq j \leqslant n\}$ be the set of roots of type A_{n-1} where

$$\boldsymbol{e}_i = (0, \cdots, 0, \underset{i}{1}, 0 \cdots, 0) \in \mathbb{Z}^n.$$

Let $\Phi^+ = \{\boldsymbol{e}_i - \boldsymbol{e}_j \mid 1 \leqslant i < j \leqslant n\}$ be the set of positive roots. There is a $\mathbb{Z}$-bilinear form $\langle -, - \rangle$ on $X = \mathbb{Z}^n$ satisfying $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \delta_{ij}$ for $1 \leqslant i,j \leqslant n$. The symmetric group $\mathfrak{S}_n$ acts on X by place permutation. The 'dot' action of $\mathfrak{S}_n$ on X is defined as: $w.\lambda = w(\lambda + \rho) - \rho$ where $\rho = (n-1, n-2, \cdots, 1, 0)$. For $\lambda \in \Lambda^+(n,r)$, let $m(\lambda)$ be the least positive integer m such that there exists an $\alpha \in \Phi^+$ with $\langle \lambda + \rho, \alpha \rangle \notin lp^m\mathbb{Z}$.

Proposition 8.1. ([3, 4]) *For $\lambda \in \Lambda^+(n,r)$, let $\mathcal{B}^{n,r}(\lambda)$ be the block of q-Schur algebras $U_k(n,r)$ containing $L_k(\lambda)$. Then, we have*

$$\mathcal{B}^{n,r}(\lambda) = (\mathfrak{S}_n.\lambda + lp^{m(\lambda)}\mathbb{Z}\Phi) \cap \Lambda^+(n,r).$$

We will denote the block of G_h containing $L_h(\lambda)$ by $\mathcal{B}^n_h(\lambda)$ for $\lambda \in \mathsf{X}_h$ and denote the block of infinitesimal q-Schur algebras $s_k(n,r)_h$ containing $\widehat{L}_1(\lambda)$ by $\mathcal{B}^{n,r}_h(\lambda)$ for $\lambda \in \mathsf{X}_h(l,r)$.

Proposition 8.2. ([3, 5]) *Assume $n = 2$. For $\lambda \in \mathsf{X}_h$, we have*

$$\mathcal{B}^2_h(\lambda) = (\mathfrak{S}_2.\lambda + lp^{m(\lambda)}\mathbb{Z}\Phi + lp^{h-1}\mathsf{X}) \cap \mathsf{X}_h.$$

For $\lambda \in \mathsf{X}_h(l,r)$ we have

$$\mathcal{B}_h^{2,r}(\lambda) = \begin{cases} (\mathfrak{S}_2.\lambda + lp^{m(\lambda)}\mathbb{Z}\Phi) \cap \mathsf{X}_h(l,r) & \text{if } m(\lambda)+1 \leqslant h, \\ \{\lambda\} & \text{if } m(\lambda)+1 > h. \end{cases}$$

For $\bar{\lambda} \in \overline{\Lambda^+(n,r)}_l$, the block of little q-Schur algebras $\tilde{u}_k(n,r)$ containing $\mathfrak{L}_k(\overline{\lambda})$ will be denoted by $\mathfrak{b}^{n,r}(\overline{\lambda})$. We now determine $\mathfrak{b}^{2,r}(\overline{\lambda})$.

Lemma 8.3. *For any $\overline{\lambda}, \overline{\mu} \in \overline{\mathsf{X}_1(l,r)}_l$, we have*

$$\mathrm{Ext}^1_{\tilde{u}_k(n,r)}(\mathfrak{L}_k(\overline{\lambda}), \mathfrak{L}_k(\overline{\mu})) \cong \mathrm{Ext}^1_{\tilde{u}_k(n,r)}(\mathfrak{L}_k(\overline{\mu}), \mathfrak{L}_k(\overline{\lambda})).$$

Proof. By [1, 3.10], there is an anti-automorphism τ on the q-Schur algebra $U_k(n,r)$ by sending $[A]$ to $[{}^tA]$ for all $A \in \Xi(n,r)$, where ${}^tA$ is the transpose of A. Since the set $\{[\![A,r]\!]_1 \mid A \in \overline{\Xi(n,r)}_l\}$ (see (4.7.1)) forms a k-basis of $\tilde{u}_k(n,r)$ by [18], we conclude that $\tau(\tilde{u}_k(n,r)) = \tilde{u}_k(n,r)$. Using τ, we may construct, for any (finite dimensional) $\tilde{u}_k(n,r)$-module M, its contravariant dual module ${}^\tau M$. Thus, as a vector space, ${}^\tau M$ is the dual space M^* of M and the action is defined by $x.f = f\tau(x)$ for all $x \in \tilde{u}_k(n,r), f \in M^*$. Since ${}^\tau(\mathfrak{L}_k(\overline{\lambda})) \cong \mathfrak{L}_k(\overline{\lambda})$ for any $\overline{\lambda} \in \overline{\mathsf{X}_1(l,r)}_l$ and $0 \to L \to M \to N \to 0$ is an exact sequence of $\tilde{u}_k(n,r)$-modules if and only if so is $0 \to {}^\tau N \to {}^\tau M \to {}^\tau L \to 0$, the result follows easily (see [31, II, 2.12(4)] for a similar result.) □

Proposition 8.4. *For $\overline{\lambda} \in \overline{\Lambda^+(2,r)}_l (= \overline{\mathsf{X}_1(l,r)}_l)$ with $\lambda \in \Lambda^+(2,r)$, if $\mathfrak{b}^{2,r}(\overline{\lambda})$ denotes the block containing $\mathfrak{L}_k(\overline{\lambda})$ for the little q-Schur algebra $\tilde{u}_k(2,r)$, then*

$$\mathfrak{b}^{2,r}(\overline{\lambda}) = \overline{(\mathfrak{S}_2.\lambda)}_l \cap \overline{\Lambda^+(2,r)}_l = \overline{\mathcal{B}^{2,r}(\lambda)}_l = \overline{\mathcal{B}_1^{2,r}(\lambda)}_l.$$

Proof. If $\mu \in \mathsf{X}_1(l,r)$ and $\mathrm{Ext}^1_{s_k(2,r)_1}(\widehat{L}_1(\lambda), \widehat{L}_1(\mu)) \neq 0$, then, by 7.1, $\mathrm{Ext}^1_{\tilde{u}_k(2,r)}(\mathfrak{L}_k(\overline{\lambda}), \mathfrak{L}_k(\overline{\mu})) \neq 0$. This proves $\overline{\mathcal{B}_1^{2,r}(\lambda)}_l \subseteq \mathfrak{b}^{2,r}(\overline{\lambda})$. Hence, 8.2 implies $\overline{(\mathfrak{S}_2.\lambda)}_l \cap \overline{\Lambda^+(2,r)}_l \subseteq \mathfrak{b}^{2,r}(\overline{\lambda})$.

On the other hand, if $\overline{\mu} \in \mathfrak{b}^{2,r}(\overline{\lambda})$ with $\mu \in \mathsf{X}_1(l,r)$ and $\mathrm{Ext}^1_{\tilde{u}_k(2,r)}(\mathfrak{L}_k(\overline{\lambda}), \mathfrak{L}_k(\overline{\mu})) \neq 0$, then there exists a $\tilde{u}_k(2,r)$-module N (and hence a G_1-module) such that $\mathrm{soc}_{\tilde{u}_k(2,r)} N = \mathfrak{L}_k(\overline{\mu})$ and $\mathrm{top}_{\tilde{u}_k(2,r)} N = \mathfrak{L}_k(\overline{\lambda})$. Equivalently, as a G_1-module, $\mathrm{soc}_{G_1} N = \mathrm{soc}_{u_k(2,r)_1} N = L_1(\mu)$ and hence $\mathrm{top}_{G_1} N = L_1(\lambda)$. So $\mathrm{Ext}^1_{G_1}(L_1(\lambda), L_1(\mu)) \neq 0$. Thus, the first assertion in 8.2 implies $\mu \in (\mathfrak{S}_2.\lambda + lp^{m(\lambda)}\mathbb{Z}\Phi + l\mathsf{X}) \cap \mathsf{X}_1$. Since $\overline{\mathsf{X}_1(l,r)}_l = \overline{\Lambda^+(2,r)}_l$, it follows from 5.5 that $\mathfrak{b}^{2,r}(\overline{\lambda}) \subseteq \overline{(\mathfrak{S}_2.\lambda)}_l \cap \overline{\Lambda^+(2,r)}_l$. Hence, $\mathfrak{b}^{2,r}(\overline{\lambda}) = \overline{(\mathfrak{S}_2.\lambda)}_l \cap \overline{\Lambda^+(2,r)}_l$, and consequently, $\mathfrak{b}^{2,r}(\overline{\lambda}) = \overline{\mathcal{B}^{2,r}(\lambda)}_l = \overline{\mathcal{B}_1^{2,r}(\lambda)}_l$, by 8.1 and 8.2. □

We are now going to establish the fact that any non-semisimple $\tilde{u}_k(2,r)$ has infinite representation type. We need the following three simple lemmas.

Lemma 8.5. *Let V be an $s_k(n,r)$-module. Then, V is an indecomposable $s_k(n,r)$-module if and only if V is an indecomposable $\tilde{u}_k(n,r)$-module.*

Proof. It is clear that V is an indecomposable $s_k(n,r)_1$-module (respectively, $u_k(n,r)_1$-module) if and only if V is an indecomposable G_1T-module (respectively, G_1-module). By [12, 3.1(18)],

V is an indecomposable G_1T-module if and only if V is an indecomposable G_1-module. The assertion now follows from (4.1.2). $\square$

Lemma 8.6 ([25, 3.4(2)]). *Let N be an $U_k(n,r)$-module with two composition factors $L_k(\lambda)$ and $L_k(\mu)$, where $\lambda \in \mathsf{X}_h$ and $\mu \in \Lambda^+(n,r)$ with $\mathrm{soc}_{U_k(n,r)} N \cong L_k(\lambda)$. Assume that $L_k(\mu) = \oplus_{j=1}^s \widehat{L}_h(\mu_j)$ is the decomposition of $L_k(\mu)$ into irreducible $s_k(n,r)_h$-modules. If $\widehat{L}_h(\lambda) \not\cong \widehat{L}_h(\mu_j)$ as G_h-modules for all j, then $\mathrm{soc}_{s_k(n,r)_h} N \cong L_k(\lambda) \cong \widehat{L}_h(\lambda)$.*

Lemma 8.7. *Let A be finite dimensional k-algebra and e is an idempotent element in A. Assume $\{L_i \mid i \in I\}$ is a complete set of non-isomorphic irreducible A-modules. Then we have*

$$Ae \cong \bigoplus_{i \in I} \dim_k(eL_i) P(L_i),$$

where $P(L_i)$ is the projective cover of L_i. In particular, if l is odd and $A = U_k(2,r)$ with $r = l$ or $l+1$, then $U_k(2,l)\mathtt{k}_{(l-1,1)} \cong P(l-1,1)$, $U_k(2,l)\mathtt{k}_{(l,0)} \cong U_k(2,l)\mathtt{k}_{(0,l)} \cong P(l,0)$, and $U_k(2,l+1)\mathtt{k}_{(l-1,2)} \cong P(l-1,2) \oplus P(l,1)$, $U_k(2,l+1)\mathtt{k}_{(l+1,0)} \oplus U_k(2,l+1)\mathtt{k}_{(1,l)} \cong 2P(l+1,0) \oplus P(l,1)$.

Proof. Since e is an idempotent element in A, Ae is projective and hence we may write

$$Ae \cong \bigoplus_{i \in I} d_i P(L_i),$$

where $d_i \in \mathbb{N}$. Then $\dim_k(eL_i) = \dim_k \mathrm{Hom}_A(Ae, L_i) = \sum_{j \in I} d_j \dim_k \mathrm{Hom}_A(P_j, L_i) = d_i$ for $i \in I$.

The last statement follows from the following facts: If $A = U_k(2,l)$, then there are $\frac{l-1}{2} + 1$ simple modules: $L_k(l-i,i)$ $(0 \leqslant i \leqslant \frac{l-1}{2})$. For each $1 \leqslant i \leqslant \frac{l-1}{2}$, $L_k(l-i,i)$ has dimension $l-2i+1$ and weights $(l-i-j, i+j)$ with $0 \leqslant j \leqslant l-2i$, while $L_k(l,0)$ has dimension 2 and weights $(l,0)$ and $(0,l)$ by the tensor product theorem. If $A = U_k(2,l+1)$, then there are $\frac{l+1}{2} + 1$ simple modules: $L_k(l+1-i,i)$ $(0 \leqslant i \leqslant \frac{l+1}{2})$. For each $1 \leqslant i \leqslant \frac{l+1}{2}$, $L_k(l+1-i,i)$ has dimension $l-2i+2$ and weights $(l+1-i-j, i+j)$ with $0 \leqslant j \leqslant l-2i+1$, while $L_k(l+1,0)$ has dimension 4 and weights $(l+1,0), (l,1), (1,l)$ and $(0,l+1)$. $\square$

For $\lambda \in \Lambda^+(n,r)$ let $P(\lambda)$ be the projective cover of $L_k(\lambda)$ as a $U_k(n,r)$-module. For $\lambda \in \Lambda^+(n,r)$ let $\mathfrak{p}(\bar{\lambda})$ be the projective cover of $\mathfrak{L}_k(\bar{\lambda})$ as a $\tilde{u}_k(n,r)$-module.

Proposition 8.8. *The algebra $\tilde{u}_k(2,l)$ has infinite representation type.*

Proof. Let $\lambda = (l,0)$ and $\mu = (l-1,1)$. By [40], the standard module $\Delta(\lambda)$ has two composition factors with socle $L_k(\mu)$. Since $U_k(2,r)$ is semisimple for $l > r$ (see, e.g., [22]), we have, for any $\nu = (\nu_1,\nu_2) \in \Lambda^+(2,l)$ with $\nu \neq \lambda$, $\Delta(\nu) \cong \Delta(\nu_1-\nu_2,0) \otimes \det_q^{\nu_2} \cong L_k(\nu_1-\nu_2,0) \otimes \det_q^{\nu_2} \cong L_k(\nu)$. Hence, by the Brauer-Humphreys reciprocity, $P(\mu) = \frac{\Delta(\mu)}{\Delta(\lambda)}$ and $P(\lambda) = \Delta(\lambda)$ are uniserial modules with composition series

$$\begin{array}{llllll} P(\mu): & L_k(\mu) & P(\lambda): & L_k(\lambda) & P(\nu): & L_k(\nu) \\ & L_k(\lambda) & & L_k(\mu) & & \\ & L_k(\mu) & & & & \end{array} \tag{8.8.1}$$

where $\nu \neq \lambda, \mu$. By 8.4, we have $\mathfrak{b}^{2,r}(\overline{\lambda}) = \{\overline{\lambda}, \overline{\mu}\}$ and $\mathfrak{b}^{2,r}(\bar{\nu}) = \{\overline{\nu}\}$ for $\nu \neq \lambda, \mu$. Using 8.7, we see that $\tilde{u}_k(2,l)\mathfrak{p}_{\overline{\mu}} = U_k(2,l)\mathtt{k}_\mu \cong P(\mu)$. Hence, $P(\mu)|_{\tilde{u}_k(2,l)}$ is a projective $\tilde{u}_k(2,l)$-module. By 8.6, we first have $\mathrm{soc}_{s_k(2,l)}\Delta(\lambda) = \widehat{L}_1(\mu)$. Applying 8.6 again to the contravariant dual of $P(\mu)/L_k(\mu)$ (see the proof of 8.3) yields $\mathrm{soc}_{s_k(2,l)}(P(\mu)/L_k(\mu)) = \widehat{L}_1(\lambda)$. Hence, $\mathrm{soc}_{s_k(2,l)}P(\mu)$ is irreducible and hence $P(\mu)|_{s_k(2,l)}$ is indecomposable. This together with 8.5 implies that $P(\mu)|_{\tilde{u}_k(2,l)}$ is indecomposable. Thus, $P(\mu)|_{\tilde{u}_k(2,l)} \cong \mathfrak{p}(\bar{\mu})$ has the following structure:

$$\begin{array}{rc} \mathfrak{p}(\bar{\mu}): & \mathfrak{L}_k(\bar{\mu}) \\ & 2\mathfrak{L}_k(\bar{\lambda}) \\ & \mathfrak{L}_k(\bar{\mu}) \end{array}$$

Here $2\mathfrak{L}_k(\bar{\lambda})$ means $\mathfrak{L}_k(\bar{\lambda}) \oplus \mathfrak{L}_k(\bar{\lambda})$. (Note that $\mathfrak{L}_k(\bar{\lambda}) \cong \mathfrak{L}_k(0,0)$.)

Now let us determine the structure of $\mathfrak{p}(\bar{\lambda})$. By 8.7 we have

$$U_k(2,l)\mathtt{k}_\lambda \cong U_k(2,l)\mathtt{k}_\delta \cong P(\lambda) \tag{8.8.2}$$

where $\delta = (0,l)$. Let

$$W_1 = \mathrm{span}\left\{ \left[\begin{pmatrix} a_{1,1} & 0 \\ a_{2,1} & 0 \end{pmatrix}\right] \middle| 1 \leqslant a_{1,1}, a_{2,1} \leqslant l-1,\ a_{1,1} + a_{2,1} = l \right\}$$

$$W_2 = \mathrm{span}\left\{ \left[\begin{pmatrix} 0 & a_{1,2} \\ 0 & a_{2,2} \end{pmatrix}\right] \middle| 1 \leqslant a_{1,2}, a_{2,2} \leqslant l-1,\ a_{1,2} + a_{2,2} = l \right\}.$$

Then, there are vector space decompositions:

$$U_k(2,l)\mathtt{k}_\lambda = W_1 \oplus \mathrm{span}\{[\left(\begin{smallmatrix} l & 0 \\ 0 & 0 \end{smallmatrix}\right)], [\left(\begin{smallmatrix} 0 & 0 \\ l & 0 \end{smallmatrix}\right)]\}, \quad U_k(2,l)\mathtt{k}_\delta = W_2 \oplus \mathrm{span}\{[\left(\begin{smallmatrix} 0 & l \\ 0 & 0 \end{smallmatrix}\right)], [\left(\begin{smallmatrix} 0 & 0 \\ 0 & l \end{smallmatrix}\right)]\}, \tag{8.8.3}$$

Clearly, $\dim_k L_k(\lambda) = 2$ and $L_k(\lambda)$ has only two weights $(l,0)$ and $(0,l)$. Thus, by (8.8.1), (8.8.2) and (8.8.3),

$$W_1 \cong W_2 \cong L_k(\mu). \tag{8.8.4}$$

Now, as a vector space, $\tilde{u}_k(2,l)\mathfrak{p}_{\overline{\lambda}} = W_1 \oplus W_2 \oplus \mathrm{span}\{\mathfrak{p}_{\overline{\lambda}}\}$. Furthermore, by 8.7, $\mathfrak{p}(\bar{\lambda}) \cong \tilde{u}_k(2,l)\mathfrak{p}_{\overline{\lambda}}$. Thus, by (8.8.4), $\mathrm{soc}_{\tilde{u}_k(2,l)}\tilde{u}_k(2,l)\mathfrak{p}_{\overline{\lambda}} = W_1 \oplus W_2 \cong 2\mathfrak{L}_k(\bar{\mu})$ and $\tilde{u}_k(2,l)\mathfrak{p}_{\overline{\lambda}}/(W_1 \oplus W_2) \cong \mathfrak{L}_k(\bar{\lambda})$. So $\mathfrak{p}(\bar{\lambda}) \cong \tilde{u}_k(2,l)\mathfrak{p}_{\overline{\lambda}}$ has the following structure:

$$\begin{array}{rc} \mathfrak{p}(\bar{\lambda}): & \mathfrak{L}_k(\bar{\lambda}) \\ & 2\mathfrak{L}_k(\bar{\mu}). \end{array}$$

Let B be the basic algebra of the block $\mathfrak{b}^{2,r}(\overline{\lambda})$ of $\tilde{u}_k(2,l)$. Let $v_0 = \mathfrak{L}_k(\bar{\lambda})$ and $v_1 = \mathfrak{L}_k(\bar{\mu})$. The Ext quiver for B is given by Figure 1:

$$\begin{tikzcd} \underset{v_0}{\bullet} \arrow[r, shift left=3, "\alpha_1"] \arrow[r, shift left=1, "\alpha_2"] & \underset{v_1}{\bullet} \arrow[l, shift left=1, "\beta_1"] \arrow[l, shift left=3, "\beta_2"'] \end{tikzcd}$$

FIGURE 1.

with relations $\beta_1\alpha_1 = \beta_2\alpha_2$ and $\beta_2\alpha_1 = \beta_1\alpha_2 = \alpha_i\beta_j = 0$ for all $i, j \in \{0,1\}$. Since $\mathfrak{p}(\bar{\mu})$ is also an injective module and $\mathfrak{p}(\bar{\mu})$ is the only indecomposable projective modules of radical length greater than 2, by [13, 9.2] the algebra B has infinite representation type if and only if B/J^2 has infinite representation type, where J is the radical of B. Thus, by applying [37, 11.8] to the quiver above, we conclude that the algebra B has infinite representation type. Hence, the algebra $\tilde{u}_k(2,l)$ has infinite representation type. □

Proposition 8.9. *The algebra $\tilde{u}_k(2,l+1)$ has infinite representation type.*

Proof. Let $\lambda = (l+1,0)$, $\mu = (l-1,2)$ and $\delta = (1,l)$. By the argument similar to the proof of 8.8 we have $P(\lambda)$ and $P(\mu)$ are uniserial modules with composition factors given by

$$P(\mu):\ \begin{matrix} L_k(\mu) \\ L_k(\lambda) \\ L_k(\mu) \end{matrix} \qquad P(\lambda):\ \begin{matrix} L_k(\lambda) \\ L_k(\mu) \\ \end{matrix} \qquad P(\nu):\ \begin{matrix} L_k(\nu) \\ \\ \end{matrix}$$

where $\nu \neq \lambda, \mu$. By 8.4, we have $\mathfrak{b}^{2,r}(\overline{\lambda}) = \{\overline{\lambda}, \overline{\mu}\}$ and $\mathfrak{b}^{2,r}(\bar{\nu}) = \{\overline{\nu}\}$ for $\nu \neq \lambda, \mu$. Applying 8.7 yields

$$\tilde{u}_k(2,l+1)\mathfrak{p}_{\overline{\mu}} = U_k(2,l+1)\mathtt{k}_\mu \cong P(\mu) \oplus P(\delta^+) \quad \text{where} \quad \delta^+ = (l,1).$$

So $P(\mu)|_{\tilde{u}_k(2,l+1)}$ is projective. A similar argument with 8.6 as in the proof of 8.8 shows that $\mathrm{soc}_{s_k(2,l+1)}P(\mu)$ is irreducible. Hence, $P(\mu)|_{s_k(2,l+1)}$ is indecomposable. Thus, by 8.5, $P(\mu)|_{\tilde{u}_k(2,l+1)}$ is an indecomposable $\tilde{u}_k(2,l+1)$-module. So, $P(\mu)|_{\tilde{u}_k(2,l+1)} \cong \mathfrak{p}(\bar{\mu})$. Now, by 7.1 and 8.6, $\mathfrak{p}(\bar{\mu})$ has the following structure:

$$\mathfrak{p}(\bar{\mu}):\ \begin{matrix} \mathfrak{L}_k(\bar{\mu}) \\ 2\mathfrak{L}_k(\bar{\lambda}). \\ \mathfrak{L}_k(\bar{\mu}) \end{matrix}$$

We now determine the structure of $V := \tilde{u}_k(2,l+1)\mathfrak{p}_{\overline{\lambda}}$. Let $W = U_k(2,l+1)\mathtt{k}_\lambda \oplus U_k(2,l+1)\mathtt{k}_\delta$. By 8.7, noting that $L_k(\delta^+)$ is the Steinberg module,

$$\mathfrak{p}(\bar{\lambda}) \oplus \mathfrak{L}_k(\bar{\delta}^+) \cong V \subseteq W \cong 2P(\lambda) \oplus L_k(\delta^+) \tag{8.9.1}$$

So, by 7.1 and 8.6, $\mathrm{soc}_{\tilde{u}_k(2,l+1)}W \cong 2\mathfrak{L}_k(\bar{\mu}) \oplus \mathfrak{L}_k(\bar{\delta}^+)$ and $W/\mathrm{soc}_{\tilde{u}_k(2,l+1)}W \cong 4\mathfrak{L}_k(\bar{\lambda})$. Thus there exist $\tilde{u}_k(2,l+1)$-submodules W_1, W_2, W_3 of W such that $\mathrm{soc}_{\tilde{u}_k(2,l+1)}W = W_1 \oplus W_2 \oplus W_3$, $W_1 \cong W_2 \cong \mathfrak{L}_k(\bar{\mu})$ and $W_3 \cong \mathfrak{L}_k(\bar{\delta}^+)$. Since $[\![\mathrm{diag}(\bar{\mu}), r]\!]_1 = [\mathrm{diag}(\mu)]$, with the notation

$$V_\mu = [\![\mathrm{diag}(\bar{\mu}), r]\!]_1 V = [\mathrm{diag}(\mu)]V \subseteq [\mathrm{diag}(\mu)]W = W_\mu,$$

one computes $\dim_k V_\mu = \dim_k W_\mu = \dim_k(\mathrm{soc}_{\tilde{u}_k(2,l+1)}W)_\mu = 3$. Thus, $V_\mu = W_\mu = (\mathrm{soc}_{\tilde{u}_k(2,l+1)}W)_\mu = (W_1)_\mu \oplus (W_2)_\mu \oplus (W_3)_\mu$. This implies $\mathrm{soc}_{\tilde{u}_k(2,l+1)}W = \tilde{u}_k(2,l+1)(W_1)_\mu \oplus \tilde{u}_k(2,l+1)(W_2)_\mu \oplus \tilde{u}_k(2,l+1)(W_3)_\mu \subseteq V$. Hence, $\mathrm{soc}_{\tilde{u}_k(2,l+1)}V = \mathrm{soc}_{\tilde{u}_k(2,l+1)}W \cong 2\mathfrak{L}_k(\bar{\mu}) \oplus \mathfrak{L}_k(\bar{\delta}^+)$. Since $\dim_k V/\mathrm{soc}_{\tilde{u}_k(2,l+1)}V = 2 = \dim_k \mathfrak{L}_k(\bar{\lambda})$ and $V/\mathrm{soc}_{\tilde{u}_k(2,l+1)}V \subseteq W/\mathrm{soc}_{\tilde{u}_k(2,l+1)}W \cong 4\mathfrak{L}_k(\bar{\lambda})$, we have $V/\mathrm{soc}_{\tilde{u}_k(2,l+1)}V \cong \mathfrak{L}_k(\bar{\lambda})$. Thus, by (8.9.1) $\mathfrak{p}(\bar{\lambda})$ has three composition factors with socle $2\mathfrak{L}_k(\bar{\mu})$.

If B denotes the basic algebra of the block $\mathfrak{b}^{2,r}(\overline{\lambda})$ of $\tilde{u}_k(2,l)$, then the computation above implies that the Ext quiver for B is the same as given in Figure 1 above with relations $\beta_1\alpha_1 = \beta_2\alpha_2$ and all other products are zero. Hence, B has infinite representation type and, consequently, $\tilde{u}_k(2,l+1)$ has infinite representation type. □

We now can establish the follow classification of finite representation type for little q-Schur algebras.

Theorem 8.10. *Assume $l' = l \geqslant 3$ is odd. The little q-Schur algebra $\tilde{u}_k(n,r) = u_k(n,r)$ has finite representation type if and only if $l > r$.*

Proof. Recall from [18, 8.2(2), 8.3] that $u_k(n,r)$ has a basis $\{[\![A,r]\!]\}_{A\in\overline{\Xi(n,r)}_l}$. If $n > 2$, then $e = \sum_{\lambda\in\overline{\Lambda(2,r)}_l}[\![\mathrm{diag}(\lambda),r]\!] \in \tilde{u}_k(n,r)$ is an idempotent and $e\tilde{u}_k(n,r)e \cong \tilde{u}_k(2,r)$. Thus, if $\tilde{u}_k(2,r)$ has infinite representation type, then so does $\tilde{u}_k(n,r)$ (see [2] or [21, I.4.7] for such a general fact). So it reduces to prove the result for $n = 2$.

If $r < l$, $\tilde{u}_k(n,r) = U_k(n,r)$ is semisimple by [22]. It remains to prove that $\tilde{u}_k(2,r)$ has infinite representation type for all $r \geq l$. By the transfer map (6.0.1), we see that either $\tilde{u}_k(2,l)$ or $\tilde{u}_k(2,l+1)$ is a homomorphic image of $\tilde{u}_k(2,r)$. Since both $\tilde{u}_k(2,l)$ and $\tilde{u}_k(2,l+1)$ have infinite representation type by 8.8 and 8.9, it follows that the algebra $\tilde{u}_k(2,r)$ and hence, $\tilde{u}_k(n,r)$, has infinite representation type for all $r \geqslant l$. □

A by-product of this result is the following determination of finite representation type of infinitesimal quantum $\mathfrak{gl}_n$.

Corollary 8.11. *The infinitesimal quantum group $u_k(n)$ has infinite representation type for any n and l. In particular, $u_k(n)$ is never semisimple.*

Proof. By 8.10 the algebra $\tilde{u}_k(n,l)$ has infinite representation type. This implies that $u_k(n)$ has infinite representation type since $\tilde{u}_k(n,l)$ is the homomorphic image of $u_k(n)$. □

9. Appendix

It is well known that $\zeta_r(\mathbf{U}(\mathfrak{sl}_n))$ is equal to $\mathbf{U}(n,r)$. In this section, we shall prove that this is also true over $\mathcal{Z}$, that is, $\zeta_r(U_{\mathcal{Z}}(\mathfrak{sl}_n)) = U_{\mathcal{Z}}(n,r)$.

Let $X_i := \{\mu \in \Lambda(n,r) \mid \max\{\mu_j - \mu_{j+1} \mid 1 \leqslant j \leqslant n-1\} = i\}$. Then we have $\Lambda(n,r) = \bigcup_{-r\leqslant i\leqslant r} X_i$ (disjoint union). The definition of $\mathbf{U}(n)$ implies the following result.

Lemma 9.1. *There is a unique $\mathbb{Q}(v)$-algebra automorphism σ on $\mathbf{U}(n)$ satisfying*

$$\sigma(E_i) = F_i,\ \sigma(F_i) = E_i,\ \sigma(K_j) = K_j^{-1}.$$

It is clear that $\sigma(\left[\begin{smallmatrix}\widetilde{K}_i;c\\ t\end{smallmatrix}\right]) = \left[\begin{smallmatrix}\widetilde{K}_i^{-1};c\\ t\end{smallmatrix}\right]$. By definition, the $\mathcal{Z}$-algebra $U_{\mathcal{Z}}(\mathfrak{sl}_n)$ is generated by the elements $E_i^{(N)}$, $F_i^{(N)}$ and $\widetilde{K}_i^{\pm1}$ $(1 \leqslant i \leqslant n,\ N \geqslant 0)$. Since $\left[\begin{smallmatrix}\widetilde{K}_i;c\\ t\end{smallmatrix}\right] \in U_{\mathcal{Z}}(\mathfrak{sl}_n)$ and $\sigma(U_{\mathcal{Z}}(\mathfrak{sl}_n)) = U_{\mathcal{Z}}(\mathfrak{sl}_n)$, we have $\left[\begin{smallmatrix}\widetilde{K}_i^{-1};c\\ t\end{smallmatrix}\right] \in U_{\mathcal{Z}}(\mathfrak{sl}_n)$. By 4.5(2), the following lemma holds in $\mathbf{U}(n,r)$.

Lemma 9.2. *Let $\lambda \in \Lambda(n,r)$. Then we have* $\left[{\widetilde{\mathtt{k}}_i;c \atop t}\right]\mathtt{k}_\lambda = \left[{\lambda_i-\lambda_{i+1}+c \atop t}\right]\mathtt{k}_\lambda$.

Theorem 9.3. *The image of $U_{\mathcal{Z}}(\mathfrak{sl}_n)$ under the homomorphism ζ_r is equal to the algebra $U_{\mathcal{Z}}(n,r)$. Hence, for any field k which is a $\mathcal{Z}$-algebra, base change induces an epimorphism $\zeta_r = \zeta_r \otimes 1 : U_k(\mathfrak{sl}_n) \to U_k(n,r)$.*

Proof. Let $U'_r = \zeta_r(U_{\mathcal{Z}}(\mathfrak{sl}_n))$. By [15], $\zeta_r(U_{\mathcal{Z}}(n)) = U_{\mathcal{Z}}(n,r)$. Hence it is enough to prove that $\mathtt{k}_\lambda \in U'_r$ for any $\lambda \in \Lambda(n,r)$. We shall prove $\mathtt{k}_\mu \in U'_r$ for any $\mu \in X_i$ by a downward induction on i.

It is clear that $X_r = \{\boldsymbol{\lambda}_i := (0,\cdots,0,\underset{i}{r},0\cdots,0) \mid 1 \leqslant i \leqslant n-1\}$ and $X_{-r} = \{\boldsymbol{\lambda}_n := (0,\cdots,0,r)\}$. By 4.5(1) and 9.2, for $1 \leqslant i \leqslant n-1$, we have

$$\begin{bmatrix} \widetilde{\mathtt{k}}_i;r \\ 2r \end{bmatrix} = \mathtt{k}_{\boldsymbol{\lambda}_i} + \sum_{\mu\in\Lambda(n,r),\,\mu\neq\boldsymbol{\lambda}_i} \begin{bmatrix} \mu_i-\mu_{i+1}+r \\ 2r \end{bmatrix}\mathtt{k}_\mu.$$

If $1 \leqslant i \leqslant n-1$, then $0 \leqslant \mu_i - \mu_{i+1} + r < 2r$ for any $\mu \in \Lambda(n,r)$ with $\mu \neq \boldsymbol{\lambda}_i$. Hence, $\left[{\mu_i-\mu_{i+1}+r \atop 2r}\right] = 0$ and $\left[{\widetilde{\mathtt{k}}_i;r \atop 2r}\right] = \mathtt{k}_{\boldsymbol{\lambda}_i} \in U'_r$ for $1 \leqslant i \leqslant n-1$. Similarly, we can prove $\left[{\widetilde{\mathtt{k}}_{n-1}^{-1};r \atop 2r}\right] = \mathtt{k}_{\boldsymbol{\lambda}_n} \in U'_r$. Hence for any $\mu \in X_r \cup X_{-r}$ we have $\mathtt{k}_\mu \in U'_r$.

Now we assume that for any $\mu \in X_j$ with $j > k$ we have $\mathtt{k}_\mu \in U'_r$. Let $\lambda \in X_k$. Then there exists some i_0 such that $\lambda_{i_0} - \lambda_{i_0+1} = k$. We need to prove $\mathtt{k}_\lambda \in U'_r$.

By 4.5(1) and 9.2, we have

$$\begin{bmatrix} \widetilde{\mathtt{k}}_{i_0};r \\ k+r \end{bmatrix} = \sum_{\mu\in X_j,\,j\neq k} \begin{bmatrix} \mu_{i_0}-\mu_{i_0+1}+r \\ k+r \end{bmatrix}\mathtt{k}_\mu + \sum_{\nu\in X_k} \begin{bmatrix} \nu_{i_0}-\nu_{i_0+1}+r \\ k+r \end{bmatrix}\mathtt{k}_\nu.$$

Note that for $j < k$ with $\mu \in X_j$, we have $0 \leqslant \mu_{i_0} - \mu_{i_0+1} + r \leqslant j + r < k + r$. Since $0 \leqslant \nu_{i_0} - \nu_{i_0+1} + r \leqslant k + r$ for $\nu \in X_k$, we have $0 \leqslant \nu_{i_0} - \nu_{i_0+1} + r < k + r$ where $\nu \in X_k$ such that $\nu_{i_0} - \nu_{i_0+1} \neq k$. It follows that

$$\begin{bmatrix} \widetilde{\mathtt{k}}_{i_0};r \\ k+r \end{bmatrix} = \sum_{\substack{\nu\in X_k \\ \nu_{i_0}-\nu_{i_0+1}=k}} \mathtt{k}_\nu + \sum_{\mu\in X_j,\,j>k} \begin{bmatrix} \mu_{i_0}-\mu_{i_0+1}+1 \\ k+1 \end{bmatrix}\mathtt{k}_\mu.$$

Let $Z := \{\nu \in X_k \mid \nu_{i_0} - \nu_{i_0+1} = k\}$. Then by induction we have

$$\sum_{\nu\in Z}\mathtt{k}_\nu \in U'_r. \tag{9.3.1}$$

For any $i \neq i_0$ and $-r \leqslant s \leqslant k$, let $Y_{s,i} := \{\nu \in Z \mid \nu_i - \nu_{i+1} = s\}$. Then for any fixed $i \neq i_0$, we have $Z = \bigcup_{-r\leqslant s\leqslant k} Y_{s,i}$ (disjointed union). Now for fixed $i \neq i_0$, we prove $\sum_{\nu\in Y_{s,i}}\mathtt{k}_\nu \in U'_r$ by induction on s.

For fixed $i \neq i_0$, let $m := \max\{s \mid Y_{s,i} \neq \varnothing,\ -r \leqslant s \leqslant k\}$. By 4.5(1) and 9.2, we have

$$\begin{aligned}\begin{bmatrix}\widetilde{\mathtt{k}}_i; r\\ m+r\end{bmatrix} &= \sum_{\mu\in\Lambda(n,r)} \begin{bmatrix}\mu_i-\mu_{i+1}+r\\ m+r\end{bmatrix}\mathtt{k}_\mu\\ &= \sum_{\nu\in Y_{m,i}}\mathtt{k}_\nu + \sum_{\substack{\nu\in Y_{s,i}\neq\varnothing\\ -r\leqslant s<m}} \begin{bmatrix}s+r\\ m+r\end{bmatrix}\mathtt{k}_\nu + \sum_{\nu\notin Z}\begin{bmatrix}\nu_i-\nu_{i+1}+r\\ m+r\end{bmatrix}\mathtt{k}_\nu\\ &= \sum_{\nu\in Y_{m,i}}\mathtt{k}_\nu + \sum_{\nu\notin Z}\begin{bmatrix}\nu_i-\nu_{i+1}+r\\ m+r\end{bmatrix}\mathtt{k}_\nu\\ &\text{(since } 0\leqslant s+r<m+r \text{ for } -r\leqslant s<m).\end{aligned}$$

Hence, multiplying both sides by $\sum_{\nu\in Z}\mathtt{k}_\nu$, (9.3.1) implies

$$\sum_{\nu\in Y_{m,i}}\mathtt{k}_\nu = \sum_{\nu\in Z}\mathtt{k}_\nu\begin{bmatrix}\widetilde{\mathtt{k}}_i; r\\ m+r\end{bmatrix}\in U'_r.$$

Now we assume $Y_{s,i}\neq\varnothing$ and for any s' such that $s'>s$ and $Y_{s',i}\neq\varnothing$ we have $\sum\limits_{\nu\in Y_{s',i}}\mathtt{k}_\nu\in U'_r$. We now prove $\sum\limits_{\nu\in Y_{s,i}}\mathtt{k}_\nu\in U'_r$.

By 4.5(1) and 9.2, we have

$$\begin{aligned}\begin{bmatrix}\widetilde{\mathtt{k}}_i; r\\ s+r\end{bmatrix} &= \sum_{\mu\in\Lambda(n,r)}\begin{bmatrix}\mu_i-\mu_{i+1}+r\\ s+r\end{bmatrix}\mathtt{k}_\mu\\ &= \sum_{\nu\in Y_{s,i}}\mathtt{k}_\nu + \sum_{\substack{\nu\in Y_{s',i}\neq\varnothing\\ s<s'\leqslant m}}\begin{bmatrix}s'+r\\ s+r\end{bmatrix}\mathtt{k}_\nu + \sum_{\substack{\nu\in Y_{s',i}\neq\varnothing\\ -r\leqslant s'<s}}\begin{bmatrix}s'+r\\ s+r\end{bmatrix}\mathtt{k}_\nu + \sum_{\nu\notin Z}\begin{bmatrix}\nu_i-\nu_{i+1}+r\\ s+r\end{bmatrix}\mathtt{k}_\nu\\ &= \sum_{\nu\in Y_{s,i}}\mathtt{k}_\nu + \sum_{\substack{\nu\in Y_{s',i}\neq\varnothing\\ s<s'\leqslant m}}\begin{bmatrix}s'+r\\ s+r\end{bmatrix}\mathtt{k}_\nu + \sum_{\nu\notin Z}\begin{bmatrix}\nu_i-\nu_{i+1}+r\\ s+r\end{bmatrix}\mathtt{k}_\nu.\end{aligned}$$

By induction we have

$$\sum_{\substack{\nu\in Y_{s',i}\neq\varnothing\\ s<s'\leqslant m}}\begin{bmatrix}s'+r\\ s+r\end{bmatrix}\mathtt{k}_\nu = \sum_{s<s'\leqslant m}\begin{bmatrix}s'+r\\ s+r\end{bmatrix}\sum_{\nu\in Y_{s',i}\neq\varnothing}\mathtt{k}_\nu\in U'_r.$$

It follows that

$$\sum_{\nu\in Y_{s,i}}\mathtt{k}_\nu + \sum_{\nu\notin Z}\begin{bmatrix}\nu_i-\nu_{i+1}+r\\ s+r\end{bmatrix}\mathtt{k}_\nu = \begin{bmatrix}\widetilde{\mathtt{k}}_i; r\\ s+r\end{bmatrix} - \sum_{\substack{\nu\in Y_{s',i}\neq\varnothing\\ s<s'\leqslant m}}\begin{bmatrix}s'+r\\ s+r\end{bmatrix}\mathtt{k}_\nu\in U'_r.$$

Hence by (9.3.1) we have

$$\sum_{\nu\in Y_{s,i}}\mathtt{k}_\nu = (\sum_{\nu\in Z}\mathtt{k}_\nu)\cdot\Big(\sum_{\nu\in Y_{s,i}}\mathtt{k}_\nu + \sum_{\nu\notin Z}\begin{bmatrix}\nu_i-\nu_{i+1}+r\\ s+r\end{bmatrix}\mathtt{k}_\nu\Big)\in U'_r.$$

Now we have proved $\sum_{\nu\in Y_{s,i}\neq\varnothing}\mathtt{k}_\nu\in U'_r$ for $i\neq i_0$ with $-r\leqslant s\leqslant k$. It is clear that we have

$$\begin{aligned}\bigcap_{\substack{i\neq i_0\\1\leqslant i\leqslant n-1}} Y_{\lambda_i-\lambda_{i+1},i} &= \{\nu\in Z\mid \nu_i-\nu_{i+1}=\lambda_i-\lambda_{i+1},\ 1\leqslant i\leqslant n-1,\ i\neq i_0\}\\ &= \{\nu\in X_k\mid \nu_i-\nu_{i+1}=\lambda_i-\lambda_{i+1},\ 1\leqslant i\leqslant n-1\}\\ &= \{\lambda\}.\end{aligned}$$

It follows that

$$\begin{aligned}\prod_{\substack{i\neq i_0\\1\leqslant i\leqslant n-1}}\sum_{\nu\in Y_{\lambda_i-\lambda_{i+1},i}}\mathtt{k}_\nu &= \sum_{\nu\in\bigcap_{i\neq i_0,\,1\leqslant i\leqslant n-1}Y_{\lambda_i-\lambda_{i+1},i}}\mathtt{k}_\nu\\ &= \mathtt{k}_\lambda\in U'_r.\end{aligned}$$

Hence, the result follows. □

Note by the proof of the above theorem that we have in fact proved $\zeta_r(U^0_{\mathcal{Z}}(\mathfrak{sl}_n))=U^0_{\mathcal{Z}}(n,r)$. It is natural to ask what is the image of $\tilde{u}_k(\mathfrak{sl}_n)$ under the map ζ_r. The following theorem answer the question.

Theorem 9.4. *If $(n,l')=1$, i.e., the integers n and l' are relatively prime, then $\zeta_r(\tilde{u}^0_k(\mathfrak{sl}_n))=\tilde{u}^0_k(n,r)$. In particular, the homomorphism $\zeta_r:\tilde{u}_k(\mathfrak{sl}_n)\to\tilde{u}_k(n,r)$ is surjective.*

Proof. Let $s=\zeta_r(\tilde{u}_k(\mathfrak{sl}_n))$, $s^+=\zeta_r(\tilde{u}^+_k(\mathfrak{sl}_n))$, $s^-=\zeta_r(\tilde{u}^-_k(\mathfrak{sl}_n))$ and $s^0=\zeta_r(\tilde{u}^0_k(\mathfrak{sl}_n))$. Then $\tilde{u}^+_k(n,r)=s^+$ and $\tilde{u}^-_k(n,r)=s^-$. Hence it is enough to prove $\mathtt{k}_i\in s^0$ for all i. Since $\mathtt{k}_1\mathtt{k}_2\cdots\mathtt{k}_n=\varepsilon^r$ by [9, 2.1], we have $\mathtt{k}_1^n=\varepsilon^r\widetilde{\mathtt{k}}_1^{n-1}\widetilde{\mathtt{k}}_2^{n-2}\cdots\widetilde{\mathtt{k}}_{n-2}^2\widetilde{\mathtt{k}}_{n-1}\in s^0$. Since $(n,l')=1$, there is some integers a,b such that $na+bl'=1$. So $\mathtt{k}_1=\mathtt{k}_1^{na+bl'}=\mathtt{k}_1^{na}\in s^0$. Then $\mathtt{k}_1^{-1}=\mathtt{k}_1^{l'-1}\in s^0$. Hence $\mathtt{k}_{i+1}^{-1}=\mathtt{k}_1^{-1}\widetilde{\mathtt{k}}_1\widetilde{\mathtt{k}}_2\cdots\widetilde{\mathtt{k}}_i\in s^0$ for all i. It follows $\mathtt{k}_{i+1}=\mathtt{k}_{i+1}^{-(l'-1)}\in s^0$ for all i. Hence the result follows. □

Remark 9.5. Note that if $(n,l')\neq 1$, the above theorem may be not true. For example, Suppose $n=l'=3=l$ and $r\geqslant 4$. Then $\widetilde{\mathtt{k}}_2=\mathtt{k}_2\mathtt{k}_3^{-1}=\varepsilon^{-r}\mathtt{k}_1\mathtt{k}_2^2$ since $\mathtt{k}_1\mathtt{k}_2\mathtt{k}_3=\varepsilon^r$. Since $\mathtt{k}_2^3=\mathtt{k}_2^l=1$, we have $\widetilde{\mathtt{k}}_1=\mathtt{k}_1\mathtt{k}_2^{-1}=\mathtt{k}_1\mathtt{k}_2^2$. Hence $\widetilde{\mathtt{k}}_2=\varepsilon^{-r}\widetilde{\mathtt{k}}_1$. It follows that $\zeta_r(\tilde{u}_k(\mathfrak{sl}_n)^0)=\mathrm{span}\{1,\widetilde{\mathtt{k}}_1,\widetilde{\mathtt{k}_1}^2\}$. So $\dim\zeta_r(\tilde{u}^0_k(\mathfrak{sl}_n))\leqslant 3$. But, by [18, 9.2], $\dim\tilde{u}^0_k(n,r)=9$. Hence, in general, $\zeta_r(\tilde{u}^0_k(\mathfrak{sl}_n))\neq\tilde{u}^0_k(n,r)$. Thus,, it is very likely that $\zeta_r:\tilde{u}_k(\mathfrak{sl}_n)\to\tilde{u}_k(n,r)$ is not surjective.

References

[1] A. A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for the quantum deformation of GL_n*, Duke Math.J. **61** (1990), 655-677.

[2] K. Bongartz, *Zykellose hlgebren sind nicht zugellos*, Representation theory II, Springer Lecture Notes in Mathematics **832** (1980), 97–102.

[3] A. G. Cox, *On some applications of infinitesimal methods to quantum groups and related algebras, Ph.D thesis*, University of London 1997.

[4] A. G. Cox, The blocks of the q-Schur algebra, J. Algebra **207** (1998), 306-325.

[5] A. G. Cox, *On the blocks of the infinitesimal Schur algebras*, Quart. J. Math **51** (2000), 39-56.

[6] R. Dipper and S. Donkin, *Quantum* GL_n, Proc. London Math. Soc. **63** (1991), no. 1, 165–211.

[7] R. Dipper and G. James, *The q-Schur algebra*, Proc.London Math.Soc. **59** (1989), 23-50.

[8] R. Dipper and G. James, *q-Tensor spaces and q-Weyl modules*, Trans. Amer. Math. Soc. **327** (1991), 251-282.

[9] S. Doty and A. Giaquinto, *Presenting Schur algebras,* International Mathematics Research Notices, **36** (2002), 1907-1944.

[10] S. R. Doty, D. K. Nakano and K. M. Peters, *On Infinitesimal Schur algebras,* Proc. London Math. Soc. **72** (1996), 588-612.

[11] S. Donkin, *On Schur algebras and related algebras IV: The blocks of the Schur algebras*, J. Algebra, **168** (1994), 400–429.

[12] S. Donkin, *The q-Schur algebra*, London Mathematical Society Lecture Note Series, **253**. Cambridge University Press, Cambridge, 1998.

[13] Y. A. Drozd and V. V. Kirichenko, *Finite-dimensional algebras*, Translated from the 1980 Russian original and with an appendix by Vlastimil Dlab. Springer-Verlag, Berlin, 1994.

[14] J. Du, *Kahzdan-Lusztig bases and isomorphism theorems for q-Schur algebras,* Contemp. Math. **139** (1992), 121-140.

[15] J. Du, *A note on the quantized Weyl reciprocity at roots of unity,* Algebra Colloq. **2** (1995), 363-372.

[16] J. Du, *q-Schur algebras, asymptotic forms and quantum* SL_n*,* J. Algebra. **177** (1995), 385-408.

[17] J. Du, *Cells in certain sets of matrices*, Töhoku Math. J. **48** (1996), 417-427.

[18] J. Du, Q. Fu and J.-p. Wang, *Infinitesimal quantum* $\mathfrak{gl}_n$ *and little q-Schur algebras,* J. Algebra **287** (2005), 199-233.

[19] J.Du and B. Parshall, *Monomial Bases For q-Schur Algebras,* Trans. Amer. Math. Soc. **355** (2003), no. 4, 1593-1620.

[20] J. Du, B. Parshall, and J.-p. Wang, *Two-parameter quantum linear groups and the hyperbolic invariance of q-Schur algebras,* J.London Math. Soc. **44**(1991), 420-436.

[21] K. Erdmann, *Blocks of tame representation type and related algebras*, Lecture Notes in Math. **1428**, Springer-Verlag, New York 1990.

[22] K. Erdmann and D. K. Nakano *Representation type of q-Schur algebras*, Trans. Amer. Math. Soc. **353** (2001), 4729-4756.

[23] Q. Fu, *A comparison of infinitesimal and little q-Schur algebras*, Comm. Algebra, **33** (2005), 2663-2682.

[24] Q. Fu, *Little q-Schur algebras at even roots of unity*, J. Algebra **311** (2007), 202-215.

[25] Q. Fu, *Finite representation type of infinitesimal q-Schur algebras*, Pacific J. Math **237** (2008), 57-76.

[26] Q. Fu, *Semisimple infinitesimal q-Schur algebras*, Arch. Math., **90** (2008), 295-303.

[27] A. M. Gainutdinov, A. M. Semikhatov, I. Yu. Tipunin, and B. L. Feigin, *Kazhdan–Lusztig correspondence for the rpresentation category of the triplet* W*-algebra in logarithmic DFT*. Theoretical and Mathematical Physics **148** (2006), 1210–1235.

[28] J. A. Green, *Polynomial Representations of* GL_n, 2nd ed., with an appendix on Schensted correspondence and Littelmann paths by K. Erdmann, J. A. Green and M. Schocker, Lecture Notes in Mathematics, no. 830, Springer-Verlag, Berlin, 2007.

[29] I. Grojnowski, *The coproduct for quantum* GL_n, preprint, 1992.

[30] H. Kondo and Y. Saito, *Indecomposable decomposition of tensor products of modules over the restricted quantum universal enveloping algebra associated to* $\mathfrak{sl}_2$, arXive: 0901.4221v3.

[31] J. C. Jantzen, Representations of Algebraic Groups, Academic Press, Boston (1987).

[32] J. C. Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics, **6**. American Mathematical Society, Providence, RI, 1996.

[33] M. Jimbo, *A q-analogue of* $U(\mathfrak{gl}(N+1))$*, Hecke algebras, and the Yang-Baxter equation,* Lett. Math. Phy. **11** (1986), 247-252.

[34] G. Lusztig, *Modular representations and quantum groups,* Comtemp. Math. **82** (1989) 59-77.

[35] G. Lusztig, *Finite dimensional Hopf algebras arising from quantized universal enveloping algebras,* J. Amer. Math. Soc. **3**(1990), 257-296.

[36] G. Lusztig, *Transfer maps for quantum affine* $\mathfrak{sl}_n$, in: Representations and quantizations (Shanghai, 1998), China Higher Educ. Press, Beijing, 2000, 341–356.

[37] R. S. Pierce, *Associative algebras*, Springer-Verlag, New York 1982.

[38] M. Takeuchi, *Some topics on* $Gl_q(n)$*,* J. Algebra **147** (1992), 379-410.

[39] M.Takeuchi, *A two-parameter quantization of GL(n)(summary),* Proc. Japan Acad. **66** (1990), 112-114.

[40] L. Thams, *The subcomodule structure of the quantum symmetric powers*, Bull. Austral. Math. Soc. **50** (1994), 29-39.

School of Mathematics, University of New South Wales, Sydney 2052, Australia.
E-mail address: j.du@unsw.edu.au *Home page:* http://web.maths.unsw.edu.au/∼jied

Department of Mathematics, Tongji University, Shanghai, 200092, China.
E-mail address: q.fu@hotmail.com

Department of Mathematics, East China Normal University, Shanghai, 200062, China.
E-mail address: jpwang@ecnu.edu.cn